Gaussian Markov random fields for MRF sampling

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Markov random fields are classical models for Bayesian image modeling [1], but their sampling relies traditionally on Gibbs techniques, implying a non-negligible computational cost. On the other hand, Gaussian Markov random fields (GMRF), which are used to model continuous and spatially-homogeneous variables, can benefit from highly efficient sampling techniques [2]. In this contribution, we propose a new discrete Markov field, based on a unit-simplex geometry, and coined Gaussian unit-simplex Markov field (GUM). The core of our proposition follows results from the three following definitions.

Definition 1 (Unit simplex). *A unit P-simplex is a regular simplex belonging in $\mathbb{R}^P$, whose $P+1$ vertices lie on a unit sphere. Let us denote $\mathbf{U}_P = \{\mathbf{v}_1, \dots, \mathbf{v}_{P+1}\}$ the set of vertices of the unit P-simplex.*

Definition 2 (Gaussian Unit-simplex Markov random field (GUM)). *Let $\mathbf{U}_{K-1}$ be the K vertices of a unit $(K-1)$-simplex (Definition 1), and $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ a GMRF taking values in $\mathbb{R}^{n(K-1)}$, such that $\mathbf{Z} = \{\mathbf{Z}_s\}_{s \in \mathcal{S}}$ and $\mathbf{Z}_s$ takes values in $\mathbb{R}^{K-1}$. We define the mapping $\phi_{K,c} \colon \mathbb{R}^{n(K-1)} \mapsto \mathbb{R}^n$ such that:*

$$\phi_{K,c}(\mathbf{Z}) = \sum_{i=1}^{K} \omega_i \pi_i^c(\mathbf{Z}) \text{ and } \forall s \in \mathcal{S},\ \pi_i^c(\mathbf{Z}_s) = \frac{\exp(-c^{-2}\|\mathbf{Z}_s - \mathbf{v}_i\|^2)}{\sum_{k=1}^{K} \exp(-c^{-2}\|\mathbf{Z}_s - \mathbf{v}_k\|^2)} \tag{1}$$

with $c > 0$, $\mathbf{v}_k, \mathbf{v}_i \in \mathbf{U}_{K-1}$ unit simplex vertices, and $\omega_i \in \Omega \subset \mathbb{N}$. $\phi_{K,c}(\mathbf{Z})$ is named a GUM random field. *π_i^c is designed to indicate, site-wise, the distance between $\mathbf{Z}$ and the $i-th$ vertices $\mathbf{v}_i$ of $\mathbf{U}_{K-1}$. We can show that $\phi_{K,c}(\mathbf{Z})$ is also Markovian.*

Definition 3 (Discrete GUM). *Let $\mathbf{Z}$ be a GMRF. From Definition 2, we have:*

$$\phi_{K,c}(\mathbf{Z}) \underset{c \to 0}{\longrightarrow} \sum_{i=1}^{K} \omega_i \delta_{\left[\|\mathbf{Z}-\mathbf{v}_i\|_2 \leq \|\mathbf{Z}-\mathbf{v}_k\|_2,\ \forall \mathbf{v}_k \in \mathbf{U}_{K-1}\right]} \tag{2}$$

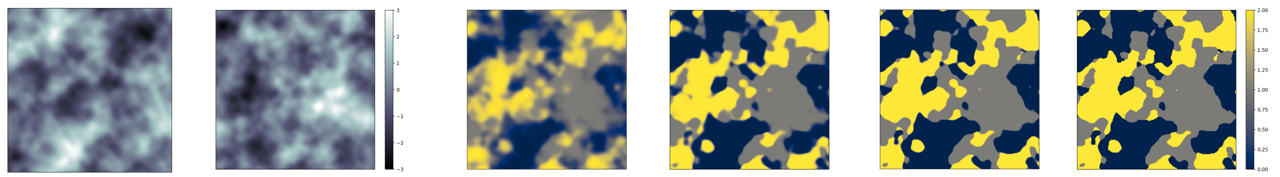
Denoting $\lim_{c \to 0} \phi_{K,c}(\mathbf{Z}) = \mathbf{X} = \{X_s\}_{s \in \mathcal{S}}$, we have $\forall s \in \mathcal{S}$:

$$X_s = \omega_{k^*} \text{ with } k^* \text{ chosen such that } \mathbf{v}_{k^*} = \underset{\mathbf{v} \in \mathbf{U}_{K-1}}{\arg\min} \|\mathbf{Z}_s - \mathbf{v}\|_2 \tag{3}$$

This discrete limit process will be referred to as a Discretized GUM *or* DGUM.

A depiction of GUM sampling is provided in Fig. 1. The computational complexity of the proposed method is that of GMRF sampling, so the proposed model can be sampled notably faster than Ising/Potts based models relying on Gibbs sampling. Our computation time is indeed improved by a factor 14 to 68 for the GPU implementation, and by a factor 200 to 290 in the CPU implementation, depending on the sample size.

Our main perspective is the development of an unsupervized inference similar based on the same models, for which the computational improvement should leverage the use of several latent field, within large images and/or 3D volumes.



(a) Realization $\mathbf{z}$: first component
(b) Realization $\mathbf{z}$: second component
(c) $\phi_{K,c}(\mathbf{z})$ with $c = 1$.
(d) $\phi_{K,c}(\mathbf{z})$ with $c = 0.5$.
(e) $\phi_{K,c}(\mathbf{z})$ with $c = 0.25$.
(f) DGUM of $\mathbf{z}$.

Figure 1: Illustration of the DGUM sampling for $K = 3$ classes, starting from the GMRF realizations (a-b), to the GUM (c-e) and its limit DGUM (f).

References.

[1] Z. Kato, J. Zerubia *et al.*, "Markov random fields in image segmentation," *Foundations and Trends® in Signal Processing*, vol. 5, no. 1–2, pp. 1–155, 2012.

[2] H. Rue and L. Held, *Gaussian Markov random fields: theory and applications.* Chapman and Hall/CRC, 2005.