Asymptotic comparison of nonparametric jump rate estimators for one-dimensional piecewise-deterministic Markov processes

Romain Azaïs and Solune Denis

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- The class is "wide enough to include as special cases virtually all the non-diffusion models of applied probability"
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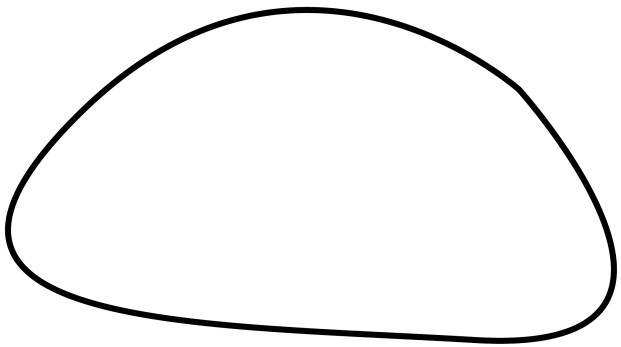

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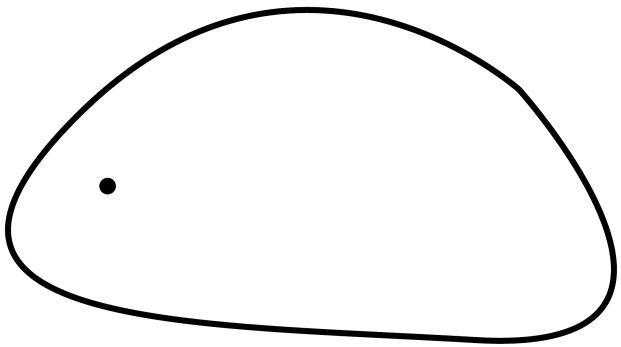

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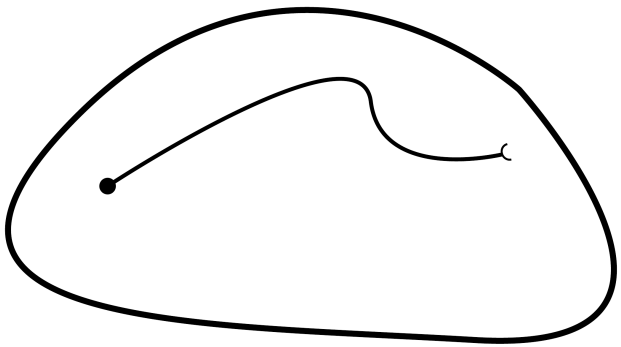

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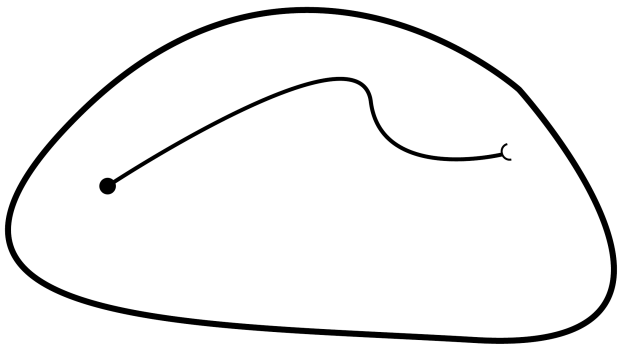

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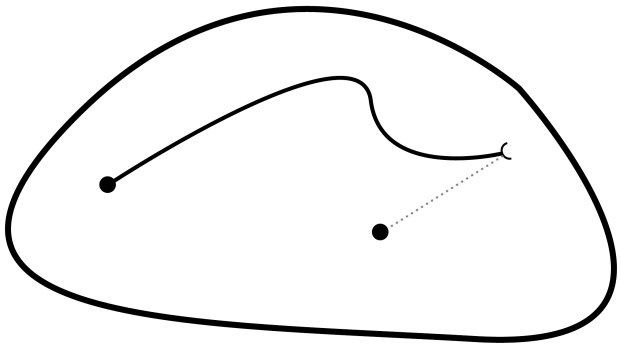

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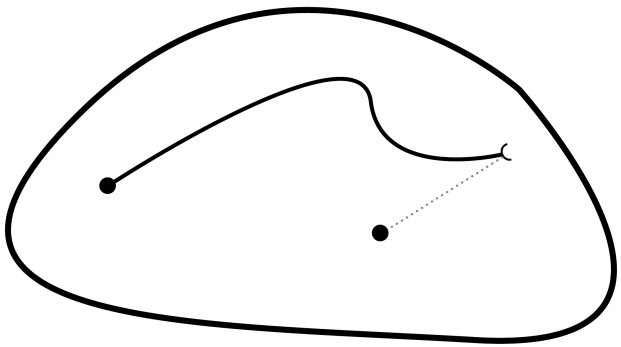

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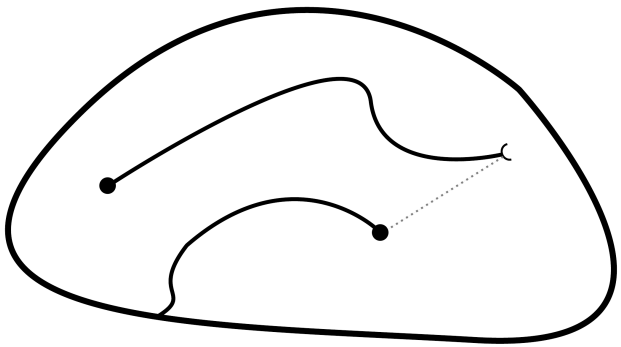

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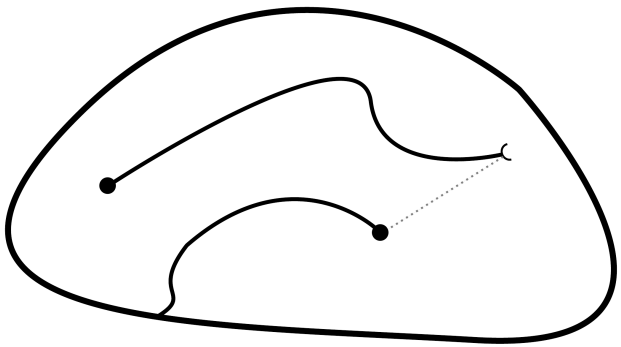

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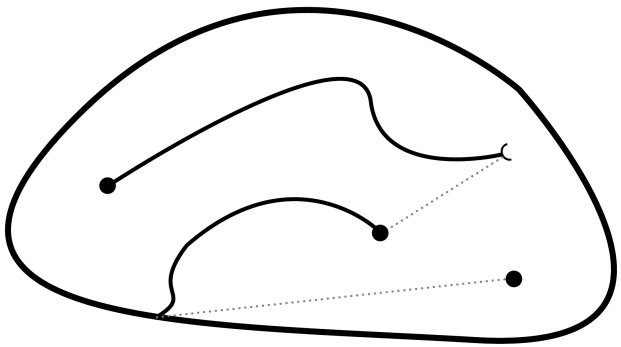

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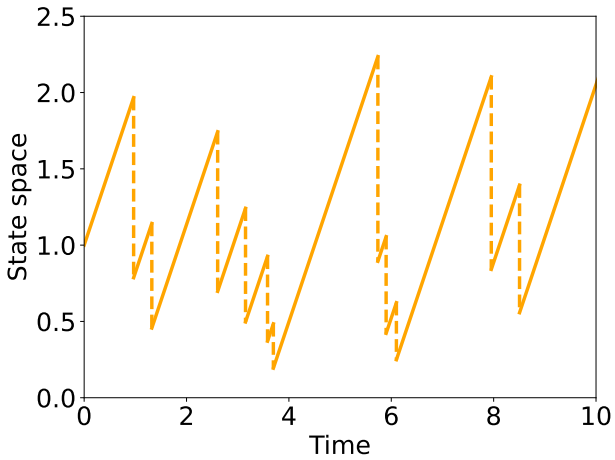
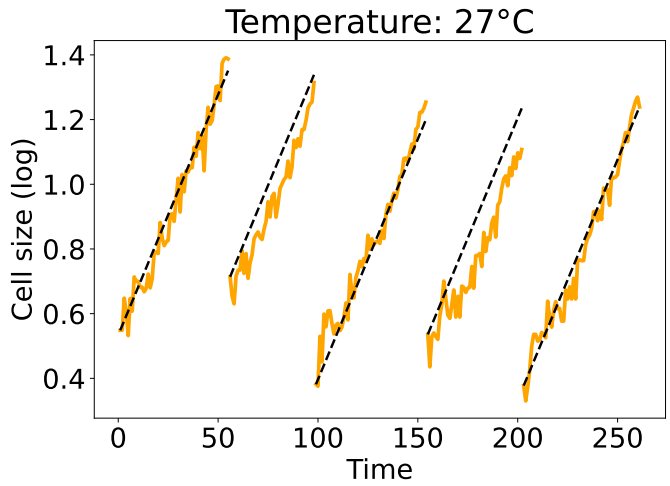
Typical PDMP in dimension 1: growth-fragmentation

- Growth: $\Phi(\cdot|x) \nearrow$, e.g. $\Phi(t|x) = x + \theta t$
- Fragmentation: $Q((0,x)|x) = 1$, e.g. $Q(\cdot|x) = \delta_{\{h(x)\}}$





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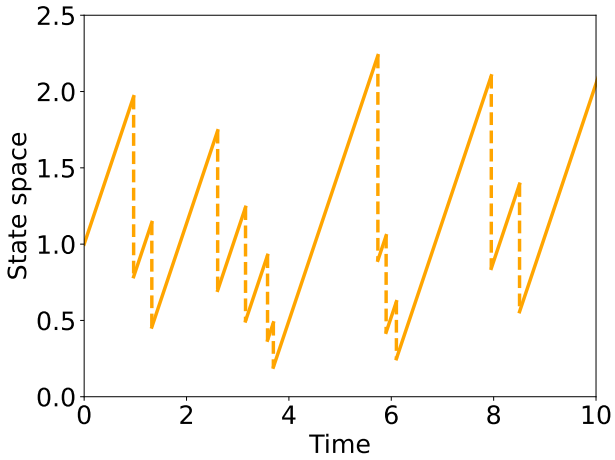

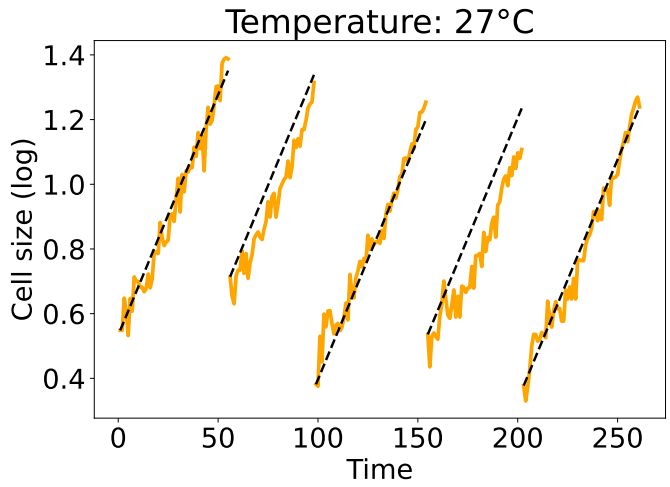

Statistical framework: one-dimensional PDMP

- Φ is known
- First n jumps (exactly) observed: $(Z_k, S_{k+1})_{0 \le k \le n}$
- Ergodicity condition: asymptotic statistics

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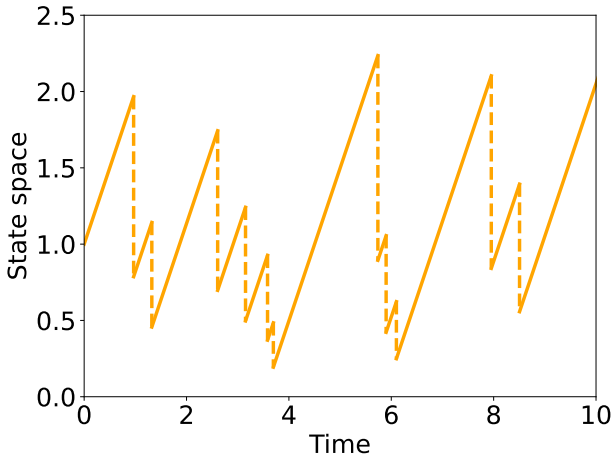

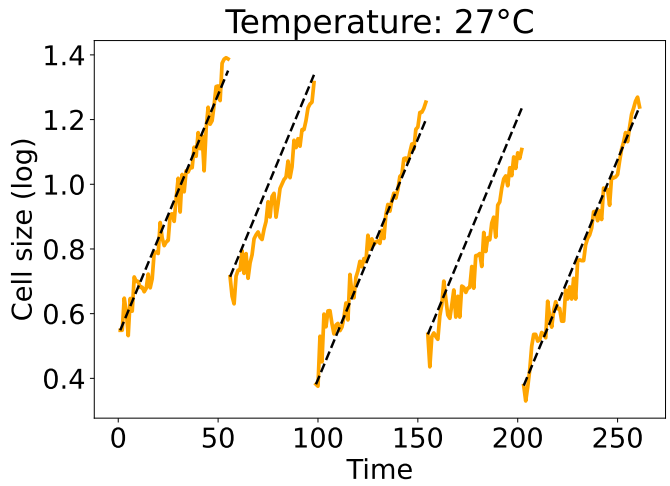

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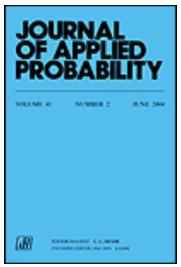
Invariant measures

$X_t \to \mu^{\mathsf{CT}}$ $\qquad$ $Z_n \to \mu$

$Z_n^- = \Phi(S_n, Z_{n-1}) \to \mu^-$

Strategies from the literature (1/2)

Fujii (2013)



- $\Phi(t|x) = x + \int_0^t \Delta(\Phi(s|x))\mathrm{d}s$
- Estimation of μ^{CT} through the local time, in relation with μ^- and λ
- μ^- estimated by kernel methods (in the (true) time of the process)
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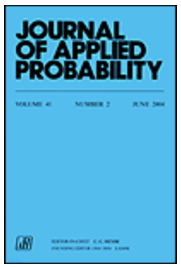

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Azaïs and Muller-Gueudin (2016)

- Multidimensional PDMP with forced jumps
- $\lambda(\Phi(t|x))$ estimated as the ratio of two invariant measures
  $\rightsquigarrow$ collection of consistent estimators of $\lambda(x)$
- Selection of the estimator with minimal asymptotic variance
- Pointwise almost sure convergence and central limit theorem

Strategies from the literature (2/2)

Krell (2016)

- $Q(\cdot|x) = \delta_{\{h(x)\}}$ assumed to be known
- $h \circ \Phi(\cdot|x)$ as a change of variable $\rightsquigarrow$ relation between λ and μ
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Krell and Schmisser (2021)

- General Q, not assumed to be known
- $\Phi(\cdot|x)$ as a change of variable $\rightsquigarrow$ relation between λ and μ^-
- μ^- estimated by projection methods
- Estimator nearly minimax (up to a $\log^2 n$ factor)

Observation 1

- Different statistical frameworks
- Different estimation methods
- Different convergence results

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Observation 2

Fujii (2013) and Krell and Schmisser (2021) estimate λ via

$$\lambda(x) = \frac{\Delta(x)\mu^{-}(x)}{\mathbb{P}_{\mu}(Z_0 \leq x < Z_1)}$$

where $\Delta(x) = \partial_t \Phi(0|x)$

Standardization (1/2)

- General one-dimensional PDMP
- Invariant measures estimated by (non-recursive) kernel methods
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- Goal: central limit theorem with explicit variance

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$$\widehat{\lambda}_n^{\clubsuit}(x) = h'(x)\Delta(x) \times \frac{\frac{1}{n}\sum_{i=0}^{n-1} K_{h_n}(Z_i - h(x))}{\frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}_{\{Z_i \leq x\}} \mathbb{1}_{\{Z_{i+1} \geq h(x)\}}}$$

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Standardization (2/2)

Azaïs and Muller-Gueudin (2016)

$$\widehat{\lambda \circ \Phi}_n(t|x) = \frac{\frac{1}{n}\sum_{i=0}^{n-1} K_{h_n^{\mathbf{s}}}(Z_i - x) K_{h_n^{\mathbf{t}}}(S_{i+1} - t)}{\frac{1}{n}\sum_{i=0}^{n-1} K_{h_n^{\mathbf{s}}}(Z_i - x) \mathbb{1}_{\{S_{i+1} > t\}}}$$

$$\mathring{\lambda}_n^{\spadesuit}(x) = \widehat{\lambda \circ \Phi}_n(\tau_x(\xi)|\xi) \quad \text{with} \quad \xi = \arg\max_{\mathcal{C}_x} \mu(\cdot) G(\tau_x(\cdot)|\cdot)$$

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Baseline

- Almost sure convergence: $\widehat{\lambda}_n^{\clubsuit}(x)$, $\widehat{\lambda}_n^{\blacklozenge}(x)$ and $\mathring{\lambda}_n^{\spadesuit}(x)$ tend to $\lambda(x)$
- Central limit theorem for $\mathring{\lambda}_n^{\spadesuit}(x)$

$$\sqrt{n h_n^{\mathsf{s}} h_n^{\mathsf{t}}} \left[\mathring{\lambda}_n^{\spadesuit}(x) - \lambda(x) \right] \to \mathcal{N}(0, \sigma_{\spadesuit}^2(x))$$

where

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Main theorem: asymptotic normality

Under ergodicity and regularity conditions,

$$\sqrt{nh_n}\left[\widehat{\lambda}_n^{\clubsuit}(x) - \lambda(x)\right] \to \mathcal{N}(0, \sigma_{\clubsuit}^2(x))$$

$$\sqrt{nh_n}\left[\widehat{\lambda}_n^{\blacklozenge}(x) - \lambda(x)\right] \to \mathcal{N}(0, \sigma_{\blacklozenge}^2(x))$$

where

$$\sigma_{\clubsuit}^2(x) = \frac{\tau^2\lambda(x)^2h'(x)}{\mu^-(x)} \quad \text{and} \quad \sigma_{\blacklozenge}^2(x) = \frac{\tau^2\lambda(x)^2}{\mu^-(x)}$$

$\widehat{\lambda}_n^{\clubsuit}(x)$ **vs.** $\widehat{\lambda}_n^{\blacklozenge}(x)$

- Same rate: $\sqrt{nh_n}$
- $\sigma^2_{\clubsuit}(x) = h'(x)\sigma^2_{\blacklozenge}(x)$

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Examples

- Linear fragmentation: $h(x) = \kappa x$, $0 < \kappa < 1$

  Then $\sigma^2_{\clubsuit}(x) = \kappa\sigma^2_{\blacklozenge}(x)$ and $\widehat{\lambda}_n^{\clubsuit}$ is uniformly better than $\widehat{\lambda}_n^{\blacklozenge}$

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- $h(x) = x/(1 + \exp(-x))$ is sublinear
  - $h'(1) < 1 \rightsquigarrow \widehat{\lambda}_n^{\clubsuit}(1)$ is better than $\widehat{\lambda}_n^{\blacklozenge}(1)$
  - $h'(2) > 1 \rightsquigarrow \widehat{\lambda}_n^{\blacklozenge}(2)$ is better than $\widehat{\lambda}_n^{\clubsuit}(2)$

Comparison with $\mathring{\lambda}_n^{\spadesuit}(x)$ is trickier

- Rate of $\mathring{\lambda}_n^{\spadesuit}(x)$: $\sqrt{nh_n^{\mathsf{s}}h_n^{\mathsf{t}}}$
- Rate of $\widehat{\lambda}_n^{\clubsuit}(x)$ and $\widehat{\lambda}_n^{\blacklozenge}(x)$: $\sqrt{nh_n}$

Extremal cases

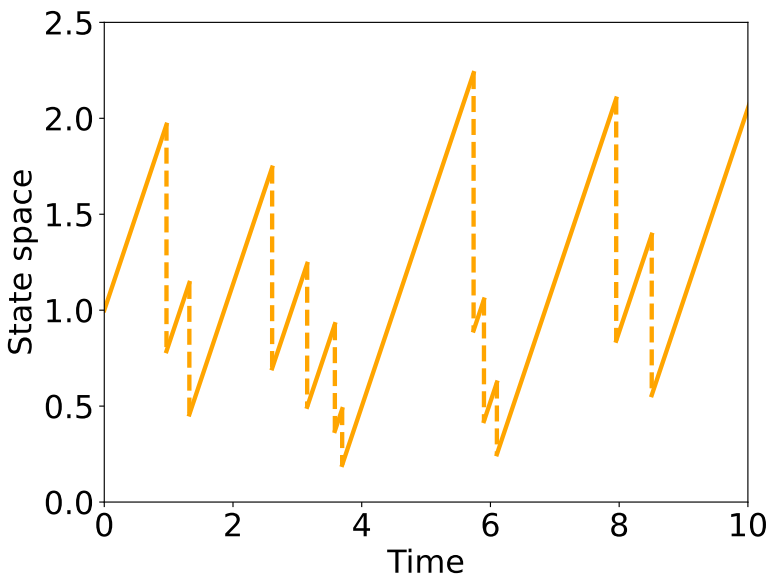
- $h_n = h_n^{\mathsf{s}}$: $\mathring{\lambda}_n^{\spadesuit}$ is asymptotically disqualified
- $h_n = h_n^{\mathsf{s}}h_n^{\mathsf{t}}$: direct comparison of variances is valid

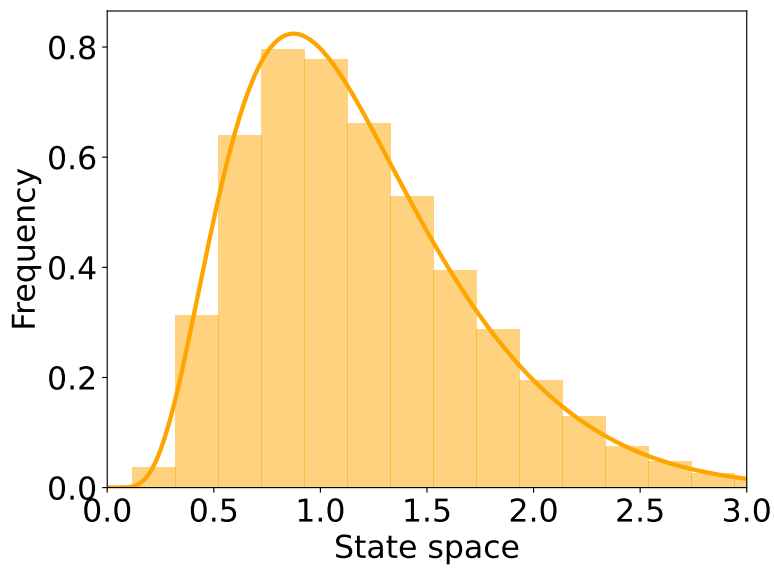
In practice, bandwidths are chosen from the available data

Growth-fragmentation model

$$\begin{cases} \lambda(x) &= x \\ \Phi(t|x) &= x + t \\ Q(\mathrm{d}y|x) &= \delta_{\{\kappa x\}}(\mathrm{d}y) \end{cases}$$

$$\mu^{\mathsf{CT}}(x) = \frac{\sqrt{2/\pi}}{\prod_{n=0}^{+\infty}(1-\kappa^{2n+1})} \sum_{n=0}^{+\infty} \frac{\kappa^{-2n}}{\prod_{k=1}^{n}(1-\kappa^{-2k})} \exp\left(-\kappa^{-2n}x^2/2\right)$$

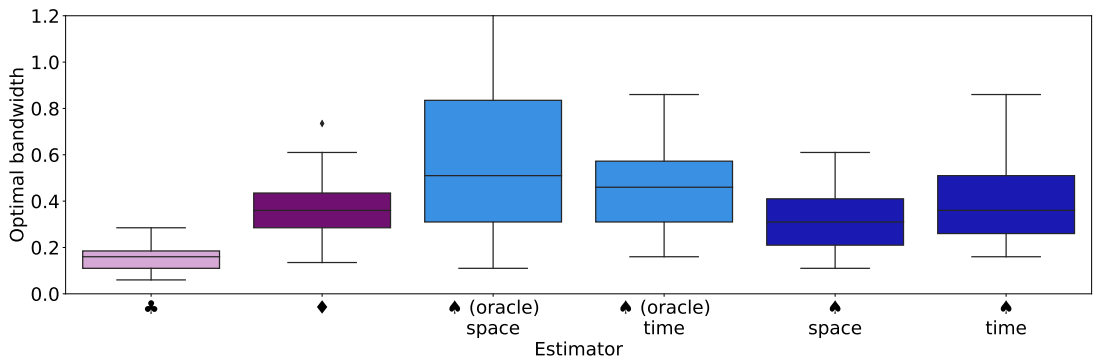


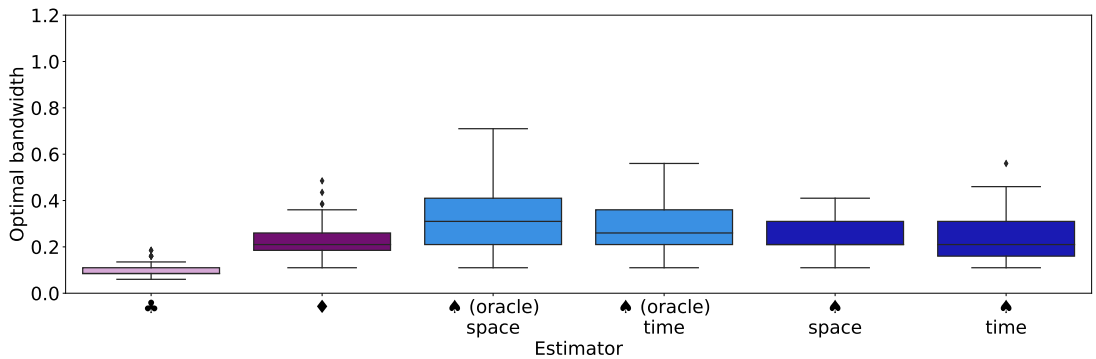


Bandwidth selection

$$h^\star = \underset{h \in \mathbb{R}_+^*}{\arg\min} \left\| \lambda - \widehat{\lambda} \right\|^2_{\mathbb{L}^2_{[0.5,2.5]}}$$

$n = 1000$





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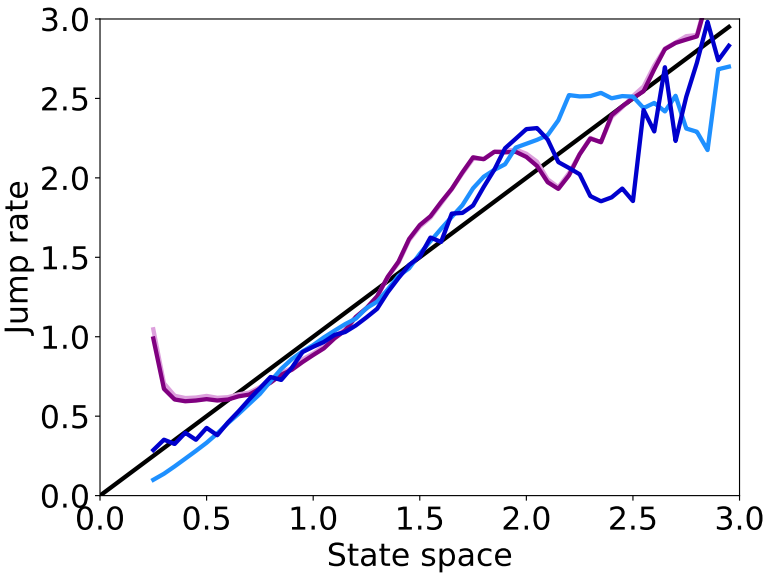
Comparison from one sample (not robust)



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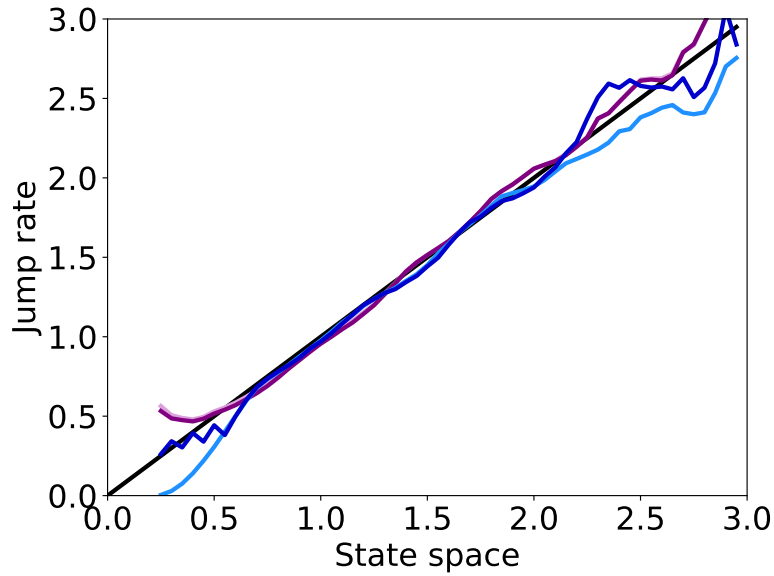
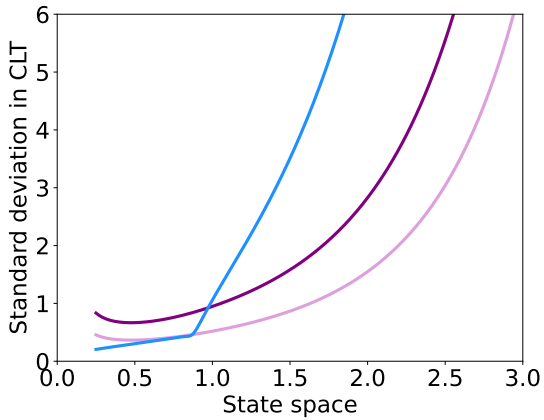


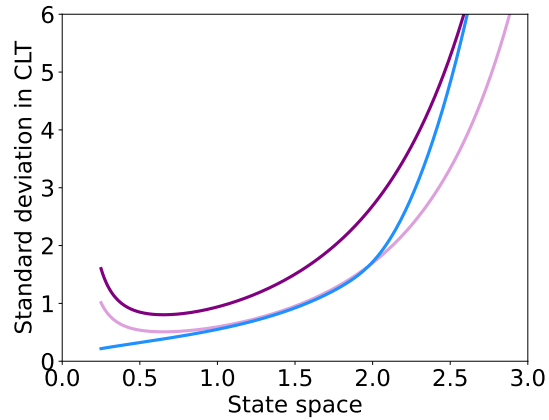
$n = 10\,000$

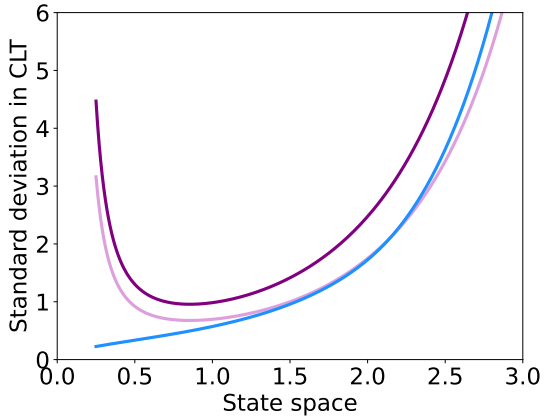
Variance from the central limit theorem (not normalized)

$\kappa = 0.3$

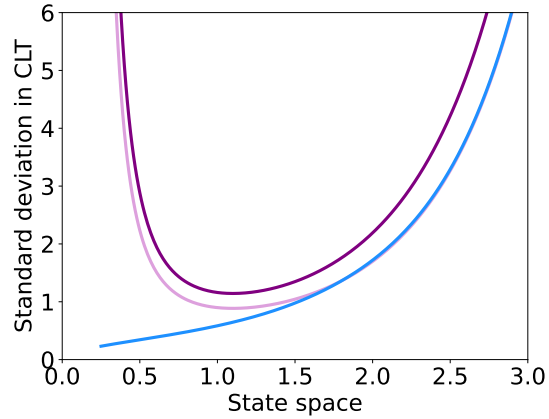


$\kappa = 0.4$





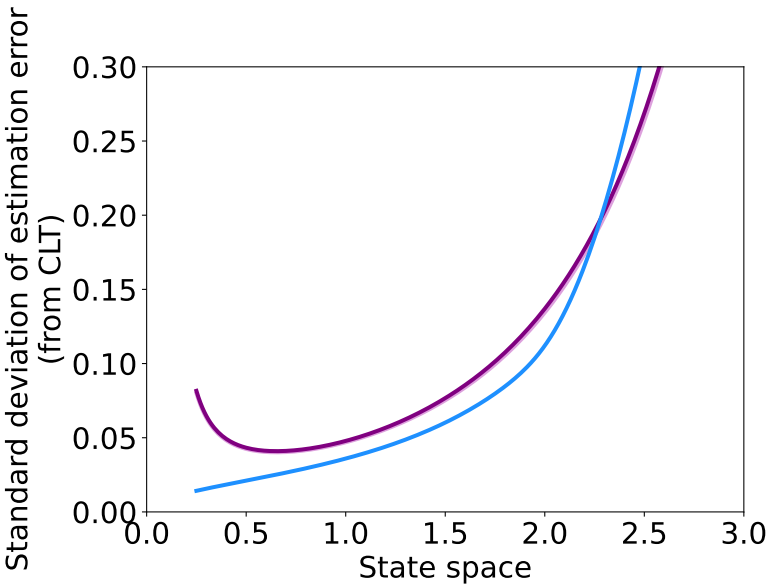
$\kappa = 0.5$



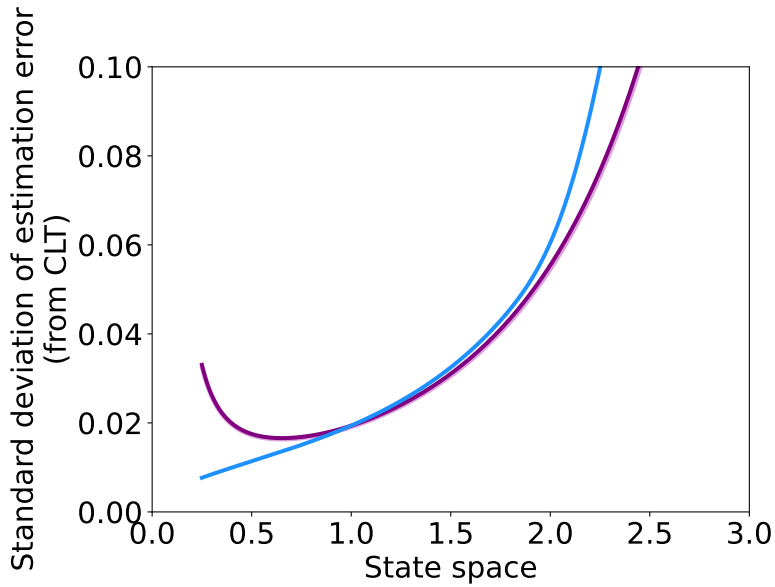
$\kappa = 0.6$

Expected asymptotic variance (normalized)

— $\kappa = 0.4$ —



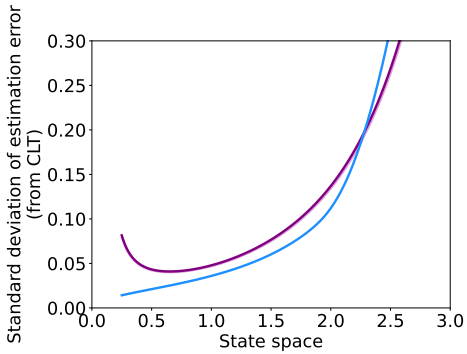
$n = 1000$

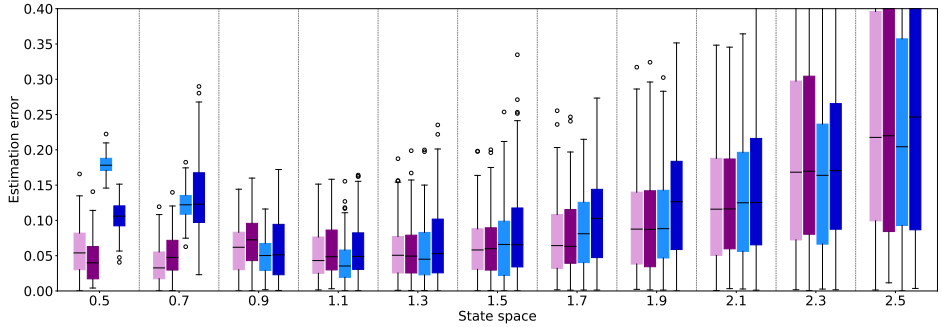


$n = 10\,000$

Expected asymptotic variance vs. estimation error

$$n = 1000$$





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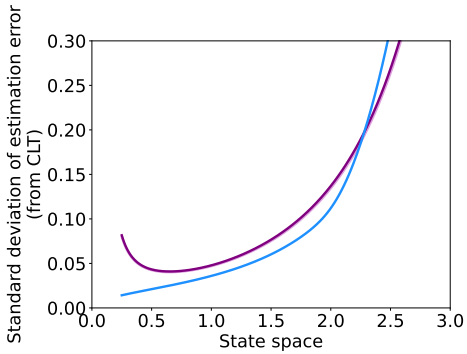

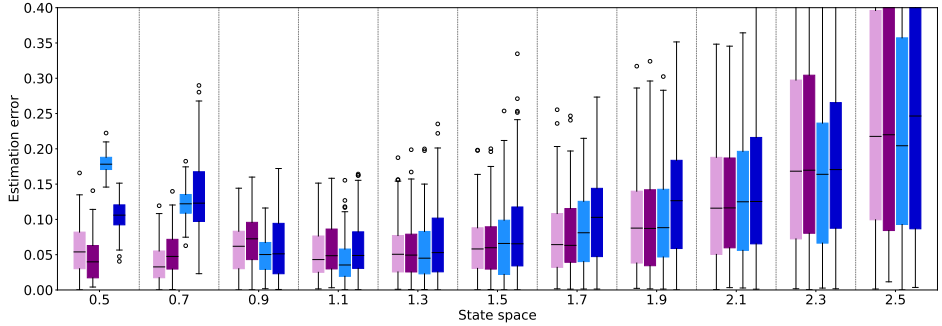

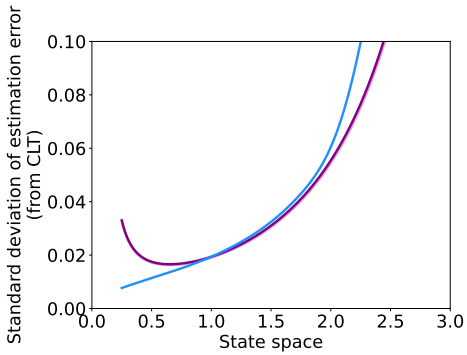

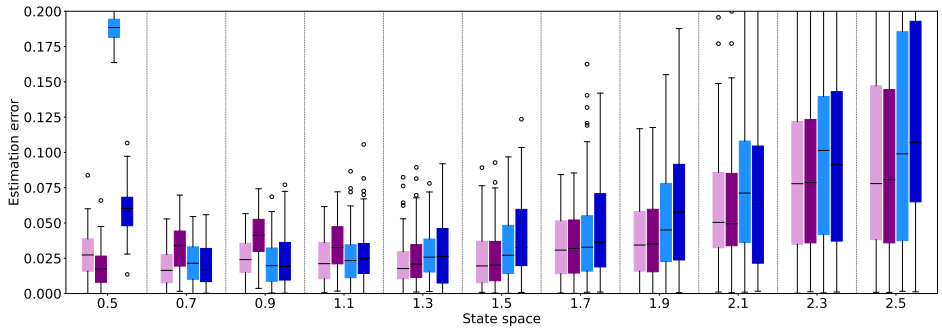

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Real data analysis (1/4)

— Data description —

Tanouchi et al. (2017): cell growth and division data (E. coli)

SCIENTIFIC DATA

Temperature		Lineages	Measurements	Division events
25°C	Total:	65	307 999	4 485
	Average:	—	4738.44	68.67
27°C	Total:	54	202 086	3 726
	Average:	—	3742.33	69
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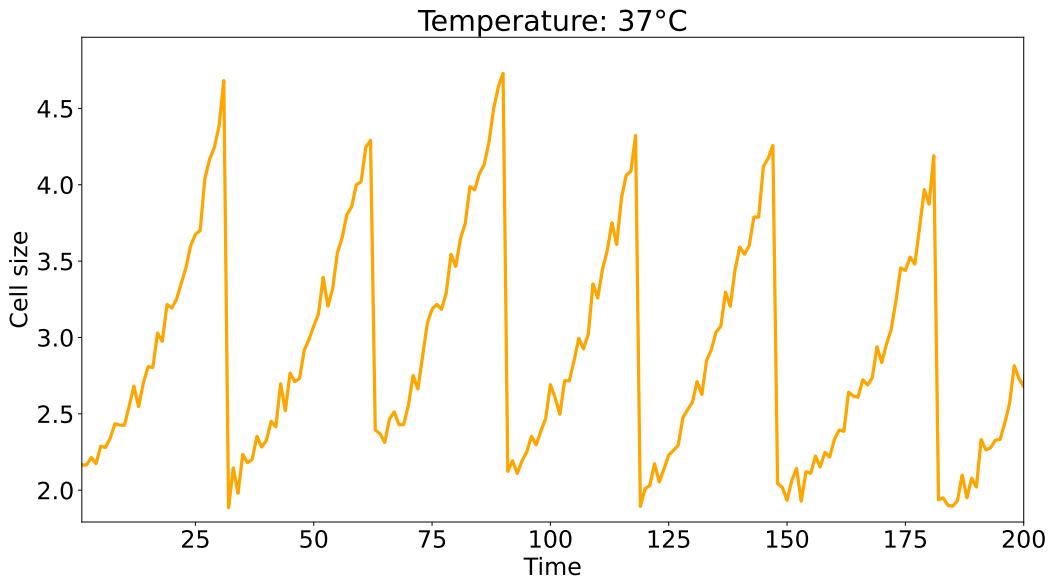

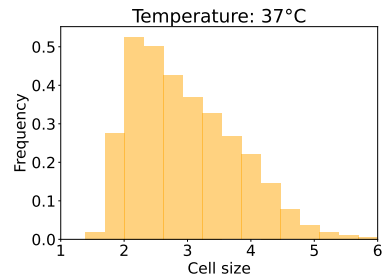

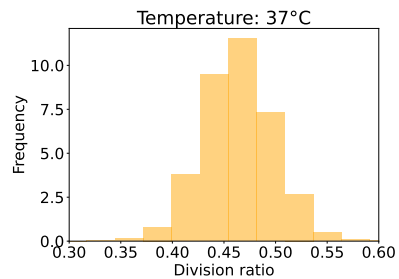

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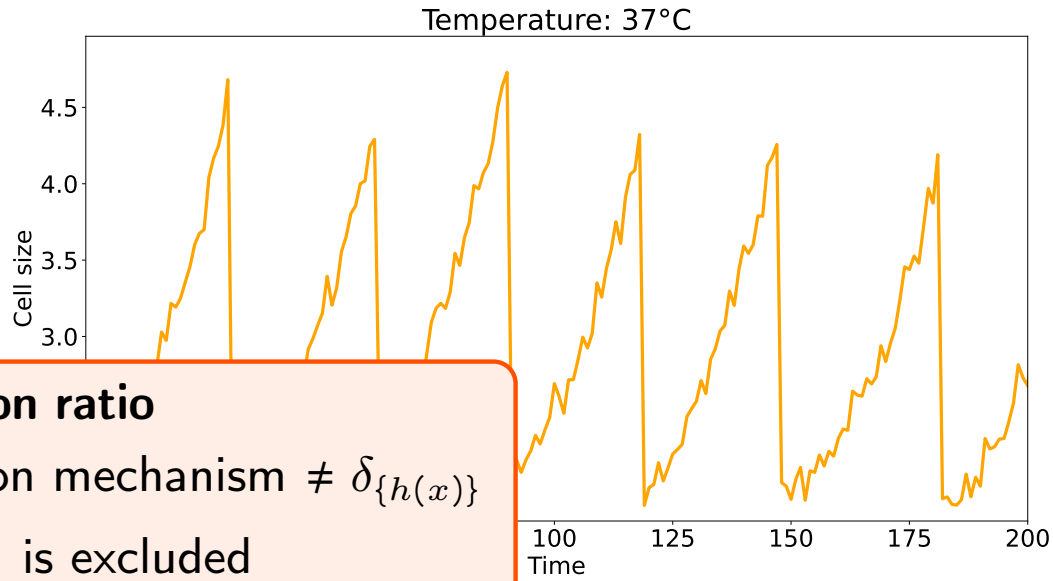
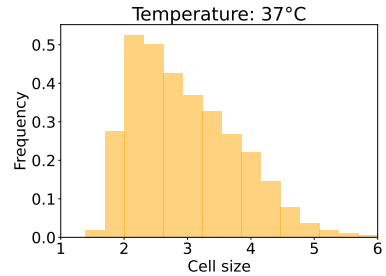
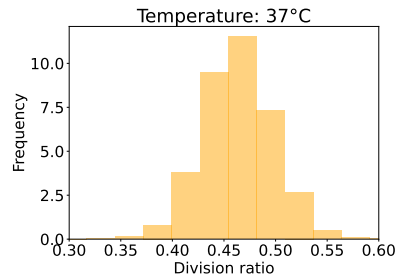






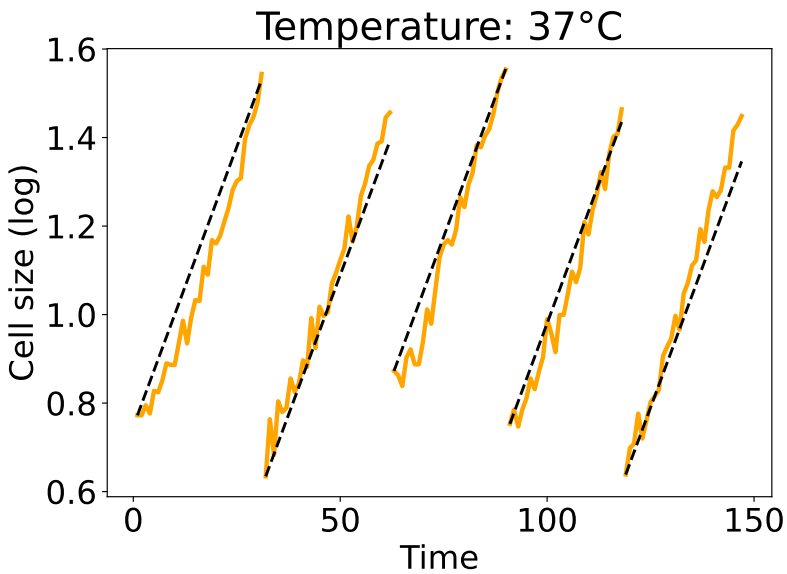
Division ratio

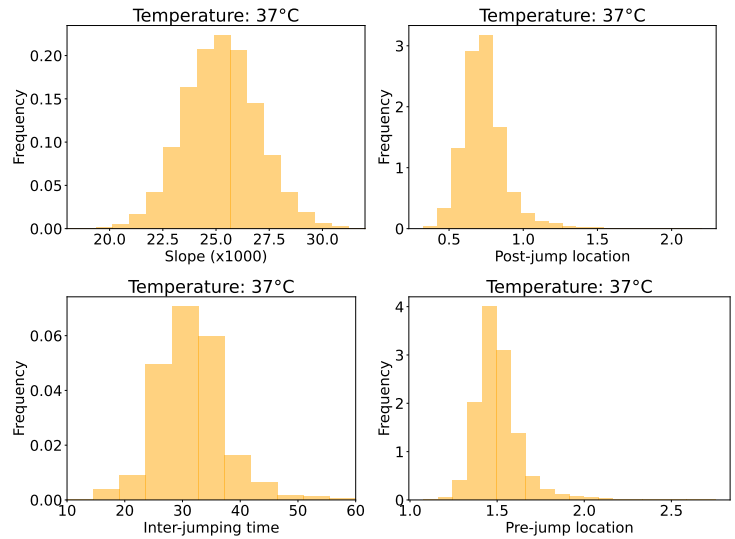
Division mechanism $\neq \delta_{\{h(x)\}}$

$\rightsquigarrow \widehat{\lambda}_n^{\clubsuit}$ is excluded

Real data analysis (2/4)

— Modeling —





- Cell growth looks linear in log scale
- Slope estimated for each of the cells

Real data analysis (2/4)

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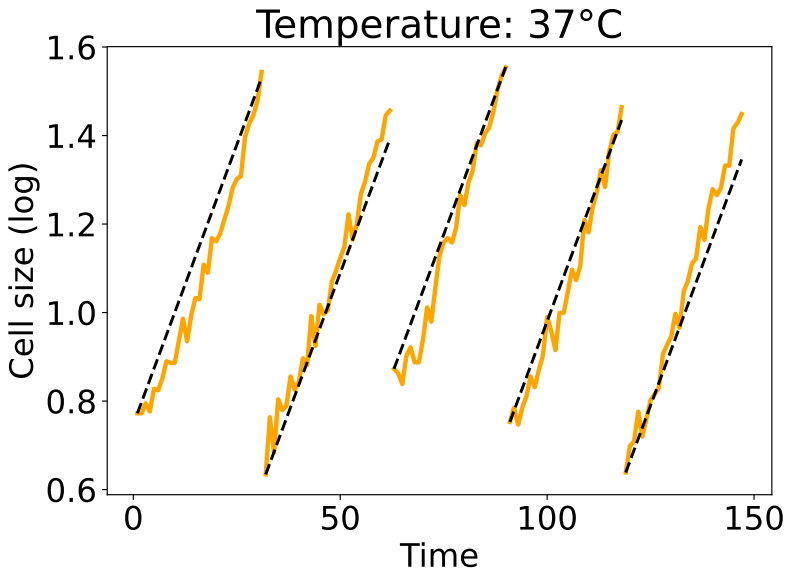
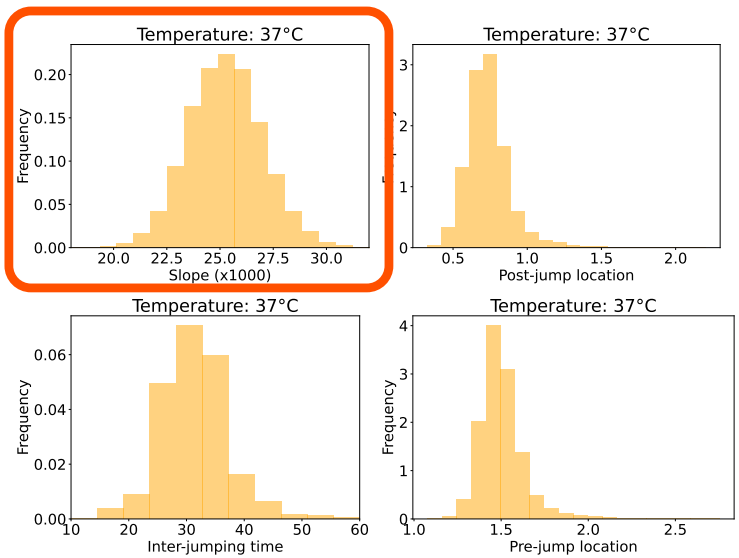




- Cell growth looks linear in log scale
- Slope estimated for each of the cells ⇝ pretty sharp distribution
- Common slope under each temperature condition: $\Phi(t|x) = x + \theta t$

Temperature	25°C	27°C	37°C
θ	0.012	0.014	0.025

Real data analysis (3/4)

— Estimation procedures —

- $\mathring{\lambda}_n^{\spadesuit}$ can not be evaluated
- Argument selection simplified for $\widehat{\lambda}_n^{\spadesuit}$: $\widehat{\lambda}_n^{\spadesuit}(x) = \widehat{\lambda \circ \Phi}_n(\tau | x - \theta\tau)$

Temperature	25°C	27°C	37°C
τ	66.6	52.4	31.6

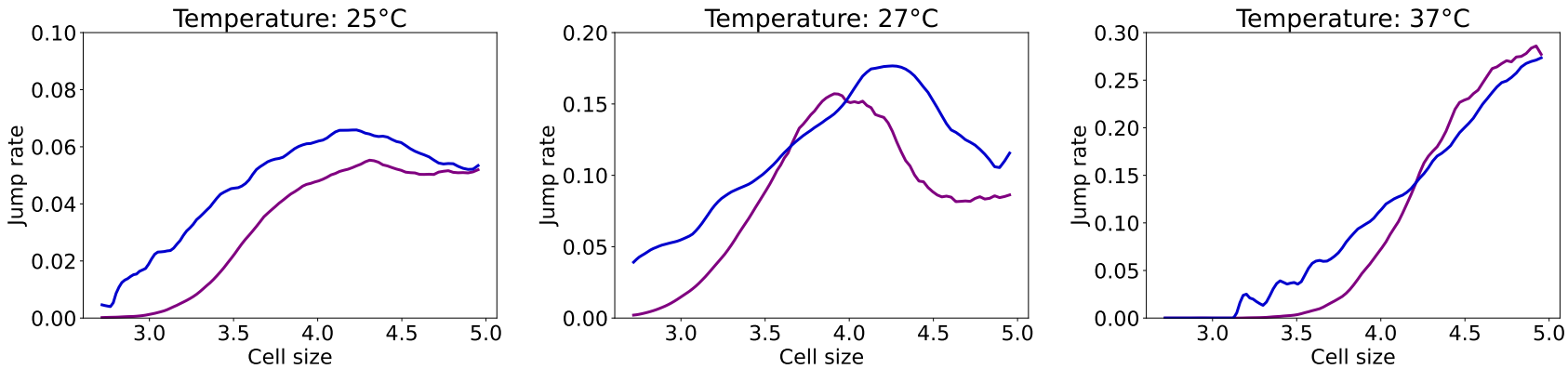
- Bandwidth selection

Temperature	Sample size	♦	♠ (space)	♠ (time)
25°C	4 485	0.05	0.06	4
27°C	3 726	0.07	0.08	8
37°C	11 040	0.02	0.03	3

Real data analysis (4/4)

— Estimation results —

$\widehat{\lambda}_n^{\blacklozenge}(\exp(x))$ vs. $\widehat{\lambda}_n^{\spadesuit}(\exp(x))$



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Azaïs and Denis (2025)

arXiv Asymptotic analysis and practical evaluation of jump rate estimators in piecewise-deterministic Markov processes

Thank you for your attention!