Limit and Ergodic Theorems for Perturbed Semi-Markov-Type Processes

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MaSeMo
1-4 July 2025, Paris

Abstract

The lecture aims to present and comment on the results of the recent books on perturbed semi-Markov-type processes and their applications:

[1] Silvestrov, D. (2025). Coupling and Ergodic Theorems for Semi-Markov-Type Processes II: Semi-Markov Processes and Multi-Alternating Regenerative Processes with Semi-Markov Modulation. Springer, Cham, xiv+550 pp.

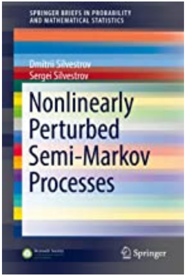
[2] Silvestrov, D. (2025). Coupling and Ergodic Theorems for Semi-Markov-Type Processes I: Markov Chains, Renewal and Regenerative Processes. Springer, Cham, xix+590 pp.

[3] Silvestrov, D. (2022). Perturbed Semi-Markov Type Processes II: Ergodic Theorems for Multi-Alternating Regenerative Processes. Springer, Cham, xvii+413 pp.

[4] Silvestrov, D. (2022). Perturbed Semi-Markov Type Processes I: Limit Theorems for Rare-Event Times and Processes. Springer, Cham, xvii+401 pp.

[5] Silvestrov, D., Silvestrov, S. (2017). Nonlinearly Perturbed Semi-Markov Processes. Springer Briefs in Probability and Mathematical Statistics, Springer, Cham, xiv+143 pp.

[5] Silvestrov, D., Silvestrov, S. (2017). Nonlinearly Perturbed Semi-Markov Processes. Springer Briefs in Probability and Mathematical Statistics, Springer, Cham, xiv+143 pp.



- Calculus of Laurent asymptotic expansions:

$$A(\varepsilon) = a_{h_A}\varepsilon^{h_A} + \cdots + a_{k_A}\varepsilon^{k_A} + o_A(\varepsilon^{k_A}) \ (-\infty < h_A < k_A < \infty),$$

where:

$$o_A(\varepsilon^{k_A})/\varepsilon^{k_A} \to 0 \ \text{ or } \ |o_A(\varepsilon^{k_A})| \leq G_A\varepsilon^{k_A+\delta_A}.$$

$\eta_\varepsilon(t), t \geq 0$ is, for $\varepsilon \geq 0$, a semi-Markov process with state space $\mathbb{X} = \{1, \ldots, m\}$ and transition probabilities $Q_{\varepsilon,ij}(t) = p_{\varepsilon,ij} F_{\varepsilon,ij}(t)$.

(α) **(a)** $p_{ij}(\varepsilon) = \sum_{l=l_{ij,-}}^{l_{ij,+}} a_{ij}[l]\varepsilon^l + o_{ij}(\varepsilon^{l_{ij,+}}), (i,j) \in A$ $\left(|o_{ij}| \leq G_{ij}\varepsilon^{m_{ij,+}[k]+\delta_{ij}}\right)$

(b) $p_{ij}(\varepsilon) = 0, (i,j) \in \bar{A}$.

$$e_{ij}(k,\varepsilon) = \int_0^\infty u^k F_{\varepsilon,ij}(du), \ k \geq 1.$$

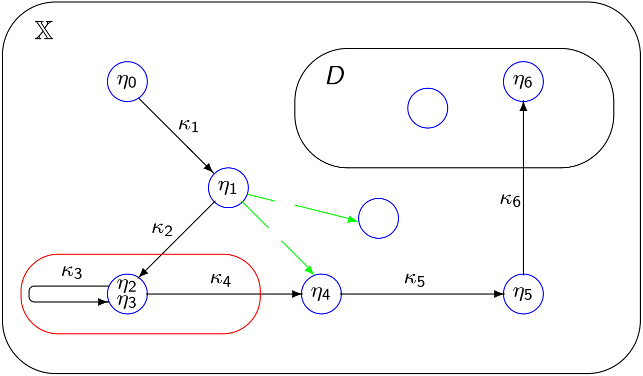
(β) **(a)** $e_{ij}(d,\varepsilon) < \infty, (i,j) \in A, \varepsilon > 0$,

(b) $e_{ij}(k,\varepsilon) = \sum_{l=m_{ij,-}[k]}^{m_{ij,+}[k]} b_{ij}[k,l]\varepsilon^l + o_{k,ij}(\varepsilon^{m_{ij,+}[k]}), (i,j) \in A, k \leq d$ $\left(|o_{k,ij}| \leq G_{k,ij}\varepsilon^{m_{ij,+}[k]+\delta_{k,ij}}\right)$.

$$\tau_\varepsilon(D) = \inf(t > 0 : \eta_\varepsilon(t) \in D), \ \text{for } D \subset \mathbb{X}.$$

- Assimptotic expansions without and with explicit upper bounds for remainders for moments of hitting times, stationary and quasi-stationary distributions.

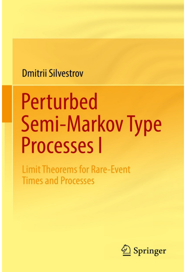
Recurrent reduction of state space

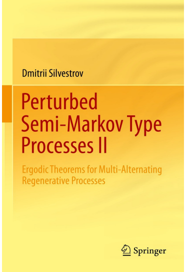


$$\mathsf{E}_i \tau_\varepsilon(D)^k = \sum_{l=m_{i,-}[k]}^{m_{i,+}[k]} c_i[k,l]\varepsilon^l + o_{k,i}(\varepsilon^{m_{i,+}[k]}), i \in \mathbb{X}, k \le d \ \left(|o_{k,i}| \le G_{k,i}\varepsilon^{m_{i,+}[k]+\delta_{i,k}}\right),$$

$$\pi_\varepsilon(i) = \sum_{l=n_{i,-}}^{n_{i,+}} \pi_i[l]\varepsilon^l + o_i(\varepsilon^{n_{i,+}}), i \in \mathbb{X} \ \left(|o_i| \le G_i \varepsilon^{m_{i,+}+\delta_i}\right).$$

- [4] Silvestrov, D.S. (2022). Perturbed Semi-Markov Type Processes I. Limit Theorems for Rare-Event Times and Processes. Springer, Cham, xvii+401 pp.
- [3] Silvestrov, D.S. (2022). Perturbed Semi-Markov Type Processes II. Ergodic Theorems for Multi-Alternating Regenerative Processes. Springer, Cham, xvii+413 pp.





$(\eta_{\varepsilon,n}, \kappa_{\varepsilon,n}, \chi_{\varepsilon,n}), n = 0, 1, \ldots$ is, for every $\varepsilon \in (0, 1]$, a Markov renewal process, i.e., a homogenous Markov chain with a phase space $\mathbb{Z} = \mathbb{X} \times [0, \infty) \times \{0, 1\}$ ($\mathbb{X} = \{1, 2, \ldots, m\}$) and transition probabilities $Q_{\varepsilon,ij}(t, \jmath) = \mathsf{P}_{(i,s,\imath)}\{\eta_{\varepsilon,1} = j, \kappa_{\varepsilon,1} \leq t, \chi_{\varepsilon,1} = \jmath\}, (i, s, \imath), (j, t, \jmath) \in \mathbb{Z}$.

$$\eta_\varepsilon(t) = \eta_{\varepsilon,n}, \text{ for } \zeta_{\varepsilon,n} \leq t < \zeta_{\varepsilon,n+1}, \ \zeta_{\varepsilon,n} = \textstyle\sum_{r=1}^{n} \kappa_{\varepsilon,r}, n = 0, 1, \ldots.$$

$$\xi_\varepsilon(t) = \sum_{1 \leq n \leq t\nu_\varepsilon} \kappa_{\varepsilon,n}, t \geq 0, \ \text{where } \nu_\varepsilon = \min(n \geq 1 : \chi_{\varepsilon,n} = 1).$$

$\eta_{\varepsilon,n}, n = 0, 1, \ldots$ is a Markov chain with transition probabilities $p_{\varepsilon,ij} = \mathsf{P}_i\{\ \eta_{\varepsilon,1} = j\} = \sum_{\jmath=0,1} Q_{\varepsilon,ij}(\infty, \jmath), \ i, j \in \mathbb{X}$.

$(\mathcal{A})$: There exists a chain of states $i_0, i_1, \ldots, i_N = i_0$ such that: (a) it contains all states from $\mathbb{X}$, (b) $\underline{\lim}_{\varepsilon \to 0} p_{\varepsilon,i_{k-1}i_k} > 0$, for $1 \leq k \leq N$.

$\pi_{\varepsilon,i}, i \in \mathbb{X}$ is the stationary distribution of the Markov chain $\eta_{\varepsilon,n}$,

$p_{\varepsilon,i} = \mathsf{P}_i\{\chi_{\varepsilon,1} = 1\} = \sum_{j \in \mathbb{X}} Q_{\varepsilon,ij}(\infty, 1), i \in \mathbb{X}$.

$(\mathcal{B})$: $0 < \max_{i \in \mathbb{X}} p_{\varepsilon,i} \to 0$ as $\varepsilon \to 0$.

$(\mathcal{C})$: $\mathsf{P}_i\{\kappa_{\varepsilon,1} > \delta / \chi_{\varepsilon,1} = 1\} \to 0$ as $\varepsilon \to 0$, for $\delta > 0$, $i \in \mathbb{X}$.

$p_\varepsilon = \sum_{i \in \mathbb{X}} p_{\varepsilon,i}\pi_{\varepsilon,i}$, $u_\varepsilon = p_\varepsilon^{-1}$,
$F_{\varepsilon,i}(t) = \mathsf{P}_i\{\kappa_{\varepsilon,1} \le t\} = \sum_{j \in \mathbb{X}, \jmath = 0,1} Q_{\varepsilon,ij}(t,\jmath), t \ \ge 0, \ i \in \mathbb{X}$,
$\theta_{\varepsilon,n}, n = 1, 2, \ldots$ are i.i.d. random variables with the distribution function $F_\varepsilon(t) = \sum_{i \in \mathbb{X}} F_{\varepsilon,i}(t)\pi_{\varepsilon,i}, t \ge 0$.

($\mathcal{D}$): $\sum_{n \le u_\varepsilon} \theta_{\varepsilon,n} \xrightarrow{\mathrm{d}} \theta_0$ as $\varepsilon \to 0$, where θ_0 is a non-zero random variable (θ_0 has an infinitely divisible distribution and $\mathsf{E}e^{-s\theta_0} = e^{-A(s)}, s \ge 0$).

T *Let model assumptions* **($\mathcal{A}$)** – **($\mathcal{C}$)** *are satisfied. Then*:
(i) *Condition* **($\mathcal{D}$)** *is necessary and sufficient for holding the following relation,* $\xi_\varepsilon(1) \xrightarrow{\mathrm{d}} \xi_0$ *as* $\varepsilon \to 0$, *where* ξ_0 *is a non-zero random variable.*
(ii) $\mathsf{E}e^{-s\xi_0} = \frac{1}{1+A(s)}, s \ge 0$.

(iii) $\xi_\varepsilon(t), t \ge 0 \xrightarrow{\mathrm{J}} \xi_0(t), t \ge 0$ *as* $\varepsilon \to 0$, *where* $\xi_0(t) = \theta_0(t\nu_0), t \ge 0$, *where:* **(a)** ν_0 *is a random variable, which has the exponential distribution with parameter* 1, **(b)** $\theta_0(t), t \ge 0$ *is a nonnegative Lévy process with the Laplace transforms* $\mathsf{E}e^{-s\theta_0(t)} = e^{-tA(s)}, s, t \ge 0$, **(c)** ν_0 *and* $\theta_0(t), t \ge 0$ *are independent.*

$$\Sigma_\varepsilon(t) = c_\varepsilon t - \sum_{n=1}^{N_\lambda(t)} \rho_n, \ \ t \geq 0,$$

where: (a) $c_\varepsilon > 0$, (b) $N_\lambda(t)$, $t \geq 0$ is a Poisson process with parameter λ, (c) $\rho_n, n = 1, 2, \ldots$ is a sequence of nonnegative i.i.d. random variables independent on the process $N_\lambda(t), t \geq 0$, (d) $P\{\rho_1 \leq u\} = H(u)$.

$$G_\varepsilon(u) = \mathsf{P}\{u + \inf_{t\geq 0} \Sigma_\varepsilon(t) \geq 0\}, \ \ u \geq 0.$$

(γ): (a) $\mu = \int_0^\infty sH(ds) < \infty$,(b) $\alpha_\varepsilon = \lambda\mu/c_\varepsilon < 1$ for $\varepsilon \in (0, 1]$ and $\alpha_\varepsilon \to 1$ as $0 < \varepsilon \to 0$.

(δ'): $\frac{t\int_t^\infty (1-H(s))ds}{\int_0^t s(1-H(s))ds} \to \frac{1-\gamma}{\gamma}$ as $t \to \infty$, for some $0 < \gamma \leq 1$.

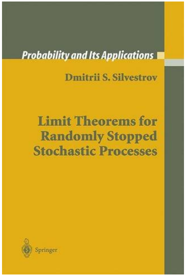
(δ''): $\frac{\int_0^{\varepsilon^{-1}} s(1-H(s))ds}{(1-\alpha_\varepsilon)\mu\varepsilon^{-1}} \to a\frac{\gamma}{\Gamma(2-\gamma)}$ as $0 < \varepsilon \to 0$, for some $a > 0$.

T *Let condition* (γ) *is satisfied. Then:* **(a)** *conditions* (δ') *and* (δ'') *are necessary and sufficient for the fulfilment of the asymptotic relation,* $G_\varepsilon(\cdot\, \varepsilon^{-1}) \Rightarrow G_0(\cdot)$ *as* $0 < \varepsilon \to 0$*, where* $G_0(\cdot)$ *is a distribution function on* $[0, \infty)$ *not concentrated at zero,* **(b)** $\int_0^\infty e^{-su} G_0(du) = \frac{1}{1+as^\gamma}, s \geq 0$.

V1: First-Rare-Event Times for Regularly Perturbed SMP

- Limit theorems for counting processes generated by flows of rare events for regularly perturbed SMP,
- Limit theorems for vector first-rare-event rewards and first-rare-event times and processes for regularly perturbed SMP with transition periods and extending phase spaces.
- Necessary and sufficient conditions for weak convergence of non-ruin distribution functions for perturbed risk processes.
- Necessary and sufficient conditions for convergence in distribution for first-rare-event times for a number of models of perturbed closed M/M-type queuing systems.
- Necessary and sufficient conditions for convergence in distribution for first-rare-event times for perturbed M/M queueing systems with bounded and unbounded queue buffers.
- Necessary and sufficient conditions for weak convergence of first hitting times for regularly perturbed SMP.

[6] Silvestrov, D.S. (2004). Limit Theorems for Randomly Stopped Stochastic Processes. Probability and Its Applications, Springer, London, xiv+398 pp.



$$(\nu_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0 \to (\nu_0(t), \xi_0(t), t \geq 0 \text{ as } \varepsilon \to 0,$$
$$\xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0 \to \xi_0(\nu_0(t)), t \geq 0 \text{ as } \varepsilon \to 0 \ ?$$

- Convergence in distribution, uniform topology U, and Skorokhod topology J for superpositions of càdlàg processes.

$(\eta_{\varepsilon,n}, \kappa_{\varepsilon,n}), n = 0, 1, \ldots$ is, for every $\varepsilon \in (0, 1]$, a Markov renewal process, i.e., a homogenous Markov chain with a phase space $\mathbb{Y}$ $= \mathbb{X} \times [0, \infty)$ ($\mathbb{X} = \{1, 2, \ldots, m\}$) and transition probabilities $Q_{\varepsilon,ij}(t) = F_{\varepsilon,ij}(t)p_{\varepsilon,ij} = \mathsf{P}_{(i,s)}\{\eta_{\varepsilon,1} = j, \kappa_{\varepsilon,1} \leq t\}$, $(i, s), (j, t) \in \mathbb{Y}$.

Semi-Markov process:

$$\eta_\varepsilon(t) = \eta_{\varepsilon,n} \text{ for } \zeta_{\varepsilon,n} \leq t < \zeta_{\varepsilon,n+1}, \text{ where } \zeta_{\varepsilon,n} = \sum_{r=1}^{n} \kappa_{\varepsilon,r}, \, n = 0, 1, \ldots,$$

$$\tau_{\varepsilon,\mathbb{D}} = \sum_{n=1}^{\nu_{\varepsilon,\mathbb{D}}} \kappa_{\varepsilon,n}, \ \text{ where } \nu_{\varepsilon,\mathbb{D}} = \min(n \geq 1 : \eta_{\varepsilon,n} \in \mathbb{D}).$$

Singularly perturbed Markov chains

$$\mathbf{P}_\varepsilon = \left[\begin{array}{cc|ccc|cc} \bullet & \bullet & \bullet & \circ & \bullet & \circ & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ \end{array}\right] \longrightarrow \mathbf{P}_0 = \left[\begin{array}{cc|ccc|cc} \bullet & \bullet & \circ & \circ & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ & \circ & \circ & \circ \\ \hline \circ & \circ & \bullet & \bullet & \bullet & \circ & \circ \\ \circ & \circ & \circ & \bullet & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \bullet & \circ & \circ & \circ \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \circ \end{array}\right] \text{ as } \varepsilon \to 0.$$

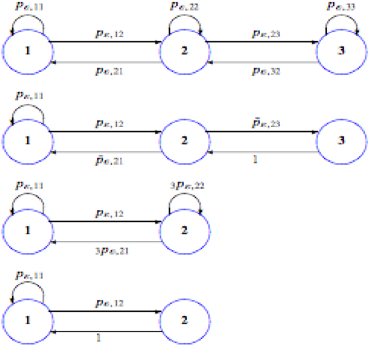
($\mathcal{E}$): $p_{\varepsilon,ij} \to p_{0,ij}$ as $\varepsilon \to 0$, for $i, j \in \mathbb{X}$

($\mathcal{F}$): $F_{\varepsilon,ij}(\cdot\, u_{\varepsilon,i}) \Rightarrow F_{0,ij}(\cdot)$ as $\varepsilon \to 0, i, j \in \mathbb{X}$, where: (a) $F_{0,ij}(\cdot)$ are distribution functions not concentrated at zero if $p_{0,ij} > 0$. (b) $u_{\varepsilon,i} \in (0, \infty)$ and $u_{\varepsilon,i} \to u_{0,i} \in (0, \infty]$ as $\varepsilon \to 0, i, j \in \mathbb{X}$.

($\mathcal{G}$): $u_{\varepsilon,i}^{-1} \int_0^\infty tF_{\varepsilon,ij}(dt) \to \int_0^\infty tF_{0,ij}(dt)$ as $\varepsilon \to 0, i, j \in \mathbb{X}$.

Recurrent algorithms of phase space reduction

$$\begin{aligned}
\eta_\varepsilon(t) &= {}_{\bar{k}_0}\eta_\varepsilon(t) \\
&\to {}_{\bar{k}_0}\tilde{\eta}_\varepsilon(t) \to {}_{\bar{k}_1}\eta_\varepsilon(t) \\
&\ldots \\
&\to {}_{\bar{k}_{h-1}}\tilde{\eta}_\varepsilon(t) \to {}_{\bar{k}_h}\eta_\varepsilon(t) \\
&\to {}_{\bar{k}_h}\tilde{\eta}_\varepsilon(t).
\end{aligned}$$



Definition *A family $\mathcal{H}$ of positive functions $h(\varepsilon)$ defined on interval $(0,1]$ is a complete family of asymptotically comparable functions if:* **(1)** *it is closed with respect to operations of summation, multiplication and division, and* **(2)** *functions $h(\cdot) \in \mathcal{H}$ have limits taking values in interval $[0,\infty]$, as $\varepsilon \to 0$.*

Examples

(a) $\frac{h(\varepsilon)}{a_h \varepsilon^{b_h}} \to 1$, **(b)** $\frac{h(\varepsilon)}{a_h \varepsilon^{b_h} e^{-c_h \varepsilon^{-1}}} \to 1$, **(c)** $\frac{h(\varepsilon)}{a_h \varepsilon^{b_h} (1 + \ln \varepsilon^{-1})^{-d_h}} \to 1$ as $\varepsilon \to 0$,

where $a_h > 0, b_h, c_h, d_h \in (-\infty, \infty)$.

($\mathcal{H}$): Non-zero transition probabilities $p_{\varepsilon,ij}$ and the initial normalisation functions $u_{\varepsilon,i}, i \in \mathbb{X}$ belong to a class $\mathcal{H}$.

Normalisation functions ${}_{\bar{k}_n}u_{\varepsilon,i}$ for the semi-Markov process ${}_{\bar{k}_n}\eta_\varepsilon(t)$ are defined for $i \in {}_{\bar{k}_n}\mathbb{X}$, $\bar{k}_n = \langle k_1, \ldots, k_n \rangle$, $k_1, \ldots, k_n \in \bar{\mathbb{D}}, 0 \leq n \leq \bar{m} - 2$ as,

$$ {}_{\bar{k}_n}u_{\varepsilon,i} = \prod_{r=0}^{n} (1 - {}_{\bar{k}_r}p_{\varepsilon,ii})^{-1} u_{\varepsilon,i} $$

($\mathcal{I}$): $\lim_{\varepsilon \to 0} \frac{{}_{\bar{k}_n}u_{\varepsilon,k_{n+1}}}{{}_{\bar{k}_n}u_{\varepsilon,i}} = w_{0,k_{n+1},i} \in [0,\infty)$, $i \in {}_{\bar{k}_n}\mathbb{X}, 0 \leq n \leq \bar{m} - 2$.

T *Let conditions* **($\mathcal{E}$) – ($\mathcal{I}$)** *are satisfied. Then, for* $i \in \bar{\mathbb{D}}$, $j \in \mathbb{D}$,

$$\mathsf{P}_i\{\tau_{\varepsilon,\mathbb{D}}/\check{u}_{\varepsilon,i} \leq \cdot, \eta_{\varepsilon,\nu_{\varepsilon,\mathbb{D}}} = j\} \Rightarrow G_{0,\mathbb{D},ij}(\cdot) \text{ as } \varepsilon \to 0,$$

where the normalisation functions $\check{u}_{\varepsilon,i}, i \in \bar{\mathbb{D}}$ *and the Laplace transforms* $\Psi_{0,\mathbb{D},ij}(s) = \int_0^\infty e^{-st} G_{0,\mathbb{D},ij}(dt), s \geq 0, i \in \bar{\mathbb{D}}, j \in \mathbb{D}$ *are given by explicit recurrent formulas.*

- Forward and backward asymptotic recurrent algorithms of phase space reduction for regularly and singularly perturbed SMP.
- Weak limit theorems for hitting and return times for regularly and singularly perturbed SMP.
- Limit theorems for expectations of hitting and return times for regularly and singularly perturbed SMP.
- Asymptotic recurrent algorithms of phase space reduction for perturbed birth-death type SMP.

$\varepsilon \in (0,1], \mathbb{X} = \{1, \ldots, m\}, \mathbb{Z}$ is an arbitrary measurable space, $\mathcal{B}_{\mathbb{Z}}$.

(1) $\bar{\xi}_{\varepsilon,i,n} = \langle \xi_{\varepsilon,i,n}(t), t \geq 0 \rangle$ is, for every $i \in \mathbb{X}$ and $n = 1, 2, \ldots$, a measurable stochastic process with a phase space $\mathbb{Z}$.

(2) $\kappa_{\varepsilon,i,n}$ is, for every $i \in \mathbb{X}$ and $n = 1, 2, \ldots$, a non-negative random variable.

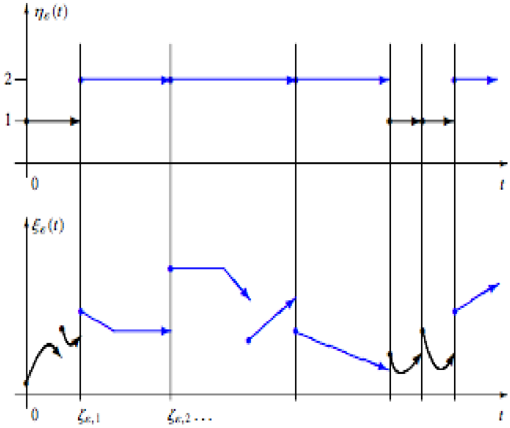
(3) $\eta_{\varepsilon,i,n}$ and η_ε be random variables taking values in the space $\mathbb{X}$, for every $i \in \mathbb{X}$ and $n = 1, 2, \ldots$.

(4) Stochastic triplets $\langle \bar{\xi}_{\varepsilon,i,n} = \langle \xi_{\varepsilon,i,n}(t), t \geq 0 \rangle, \kappa_{\varepsilon,i,n}, \eta_{\varepsilon,i,n} \rangle$, $i \in \mathbb{X}$, $n = 1, 2, \ldots$ and the random variable η_ε are mutually independent.

(5) Joint distributions of random variables $\xi_{\varepsilon,i,n}(t_k), k = 1, \ldots, r,\ \ \kappa_{\varepsilon,i,n}$, and $\eta_{\varepsilon,i,n}$ do not depend on $n \geq 1$, for every $i \in \mathbb{X}$ and $t_k \geq 0, k = 1, \ldots, r, r \geq 1$.

$$\begin{cases} \eta_{\varepsilon,n} = \eta_{\varepsilon,\eta_{\varepsilon,n-1},n}, n = 1, 2, \ldots, \eta_{\varepsilon,0} = \eta_\varepsilon, \\ \zeta_{\varepsilon,n} = \kappa_{\varepsilon,\eta_{\varepsilon,0},1} + \cdots + \kappa_{\varepsilon,\eta_{\varepsilon,n-1},n}, n = 1, 2, \ldots, \zeta_{\varepsilon,0} = 0, \\ \xi_\varepsilon(t) = \xi_{\varepsilon,\eta_{\varepsilon,n-1},n}(t - \zeta_{\varepsilon,n-1}), \eta_\varepsilon(t) = \eta_{\varepsilon,n-1}, \text{for}\, \zeta_{\varepsilon,n-1} \leq t < \zeta_{\varepsilon,n}, n \geq 1. \end{cases}$$

$(\xi_\varepsilon(t), \eta_\varepsilon(t)), t \geq 0$ is a **regenerative process**, if $m = 1$; an **alternating regenerative process**, if $m = 2$; a **multi-alternating regenerative process**, if $m > 2$; $\eta_\varepsilon(t), t \geq 0$ is a **modulating semi-Markov process**.



$P_{\varepsilon,ij}(t, A) = \mathsf{P}_i\{\xi_\varepsilon(t) \in A, \eta_\varepsilon(t) = j\}, \ t \geq 0, A \in \mathcal{B}_{\mathbb{Z}}, i, j \in \mathbb{X}.$

$$\{P_{\varepsilon,ij}(t, A) = \mathrm{I}(i = j)q_{\varepsilon,i}(t, A) + \sum_{k\in\mathbb{X}} \int_0^t P_{\varepsilon,kj}(t - s, A)Q_{\varepsilon,ik}(ds), t \geq 0, i \in \mathbb{X},$$

where $Q_{\varepsilon,ik}(t) = \mathsf{P}_i\{\zeta_{\varepsilon,1} \leq t, \eta_{\varepsilon,1} = k\}, \ q_{\varepsilon,i}(t, A) = \mathsf{P}_i\{\xi_\varepsilon(t) \in A, \zeta_{\varepsilon,1} > t\}.$

$(\mathcal{J})$: $p_{\varepsilon,ij} \to p_{0,ij}$ as $\varepsilon \to 0$, for $i, j \in \mathbb{X} = \{1, 2\}$.

$(\mathcal{K})$: $F_{\varepsilon,ij}(\cdot) \Rightarrow F_{0,ij}(\cdot)$ as $\varepsilon \to 0, i, j \in \mathbb{X}$, where $F_{0,ij}(\cdot)$ are a non-arithmetic distribution functions, for $i, j \in \mathbb{X}$ such that $p_{0,ij} > 0$.

$(\mathcal{L})$: $e_{\varepsilon,ij} = \int_0^\infty tF_{\varepsilon,ij}(dt) \to e_{0,ij} = \int_0^\infty tF_{0,ij}(dt)$ as $\varepsilon \to 0$, for $i, j \in \mathbb{X}$.

$(\mathcal{M})$: $q_{\varepsilon,i}(s_\varepsilon, A) \to q_{0,i}(s, A)$ for any $0 \leq s_\varepsilon \to s$, as $\varepsilon \to 0$, and $s \in \mathrm{U}_A$, $A \in \Gamma$, $i \in \mathbb{X}$ where: (a) U_A is some Borel subset of $[0, \infty)$ such that the Lebesgue measure $m(\bar{\mathrm{U}}_A) = 0$, (b) $q_{0,i}(s, A)$ is a measurable function continuous almost everywhere with respect to the Lebesgue measure on $[0, \infty)$, (c) $\Gamma \subseteq \mathcal{B}_\mathbb{Z}$ and $\mathbb{Z} \in \Gamma$.

Condition **$(\mathcal{M})$** implies that Γ is closed with respect to the operation of union for not intersecting sets, the operation of difference for sets connected by the relation of inclusion, and the complement operation.

$(\mathcal{N})$: $p_{0,12} \vee p_{0,21} > 0$ (regularly perturbed ARP).

$(\mathcal{O})$: $0 < p_{\varepsilon,12}, p_{\varepsilon,21} \to 0$ as $\varepsilon \to 0$ (singularly perturbed ARP).

$(\mathcal{P})$: $0 < p_{\varepsilon,12} \to 0$ as $\varepsilon \to 0$, $p_{\varepsilon,21} \equiv 0$ or $0 < p_{\varepsilon,21} \to 0$ as $\varepsilon \to 0$, $p_{\varepsilon,12} \equiv 0$ (super-singularly perturbed ARP).

($\mathcal{Q}$): $p_{\varepsilon,12}/p_{\varepsilon,21} \to \beta \in [0, \infty]$ as $\varepsilon \to 0$.

The individual ergodic theorems for singularly perturbed models:

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi_{ij}^{(\beta)}(t, A) \text{ as } \varepsilon \to 0, \tag{1}$$

which hold for any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ such that,

$$t_\varepsilon/v_\varepsilon \to t \in [0, \infty] \text{ as } \varepsilon \to 0, \text{ or } t_\varepsilon/w_\varepsilon \to t \in [0, \infty] \text{ as } \varepsilon \to 0,$$

where v_ε and w_ε are so-called time compression factors,

$$v_\varepsilon = p_{\varepsilon,12}^{-1} + p_{\varepsilon,21}^{-1} \geq w_\varepsilon = (p_{\varepsilon,12} + p_{\varepsilon,21})^{-1}.$$

($\mathcal{Q}$) implies that $w_\varepsilon/v_\varepsilon \to \beta/(1+\beta^2)$ as $\varepsilon \to 0$. Obviously, $w_\varepsilon = O(v_\varepsilon)$ as $\varepsilon \to 0$, if $\beta \in (0, \infty)$, while $w_\varepsilon = o(v_\varepsilon)$ as $\varepsilon \to 0$, if $\beta = 0$ or $\beta = \infty$.

$$t_\varepsilon \prec w_\varepsilon \qquad \underset{\vee}{t_\varepsilon \sim w_\varepsilon} \qquad w_\varepsilon \prec t_\varepsilon \prec v_\varepsilon \qquad \underset{\vee}{t_\varepsilon \sim v_\varepsilon} \qquad v_\varepsilon \prec t_\varepsilon$$

We refer to the ergodic relation (1) as **short**-time -, **long-**, or **super-long** ergodic theorems if, respectively $t = 0$, $t \in (0, \infty)$, or $t = \infty$.

T *Let conditions* $(\mathcal{J})$ – $(\mathcal{M})$, $(\mathcal{O})$, *and* $(\mathcal{Q})$ (*with some* $\beta \in (0, \infty)$) *are satisfied. Then the following ergodic relation takes place for any* $A \in \Gamma$, $i, j \in \mathbb{X}$ *and any* $0 \leq t_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *such that* $t_\varepsilon / v_\varepsilon \to t \in (0, \infty)$ *as* $\varepsilon \to 0$,

$$P_{\varepsilon,ij}(t_\varepsilon, A) \to \pi_{ij}^{(\beta)}(t, A) = p_{ij}^{(\beta)}(t)\pi_{0,j}(A) \text{ as } \varepsilon \to 0.$$

where: **(a)** $p_{ij}^{(\beta)}(t), t \geq 0, i, j \in \mathbb{X}$ be transition probabilities for continuous time, homogeneous Markov chain, with phase space $\mathbb{X} = \{1, 2\}$ transition intensities $\lambda_{12} = (1 + \beta)/e_{0,1}$, $\lambda_{21} = (1 + \beta^{-1})/e_{0,2}$, where $e_{0,i} = \sum_{j \in \mathbb{X}} e_{0,ij} p_{0,ij}, i \in \mathbb{X}$, and $p_{ij}^{(\beta)}(t), t \geq 0, i, j \in \mathbb{X}$ be transition probabilities for this Markov chain, **(b)** $\pi_{0,j}(A) = \frac{1}{e_{0,j}} \int_0^\infty q_{0,j}(s) ds$, $A \in \mathcal{B}_{\mathbb{Z}}, j \in \mathbb{X}$.

- The complete classification of super-long-, long- and short-time ergodic theorems for regularly, singularly, and super-singularly perturbed alternating regenerative processes (based on 26 ergodic theorems) is given.

[7] Gyllenberg, M., Silvestrov, D.S. (2008). Quasi-Stationary Phenomena in Nonlinearly Perturbed Stochastic Systems. De Gruyter Expositions in Mathematics, **44**, Walter de Gruyter, Berlin, ix+579 pp.

$$x_\varepsilon(t) = q_\varepsilon(t) + \int_0^t x_\varepsilon(t-s)F_\varepsilon(ds), \; t \geq 0.$$

$$x_\varepsilon(t) \to ? \text{ as } t \to \infty \text{ and } \varepsilon \to 0.$$

- Renewal theorem for perturbed renewal equation.

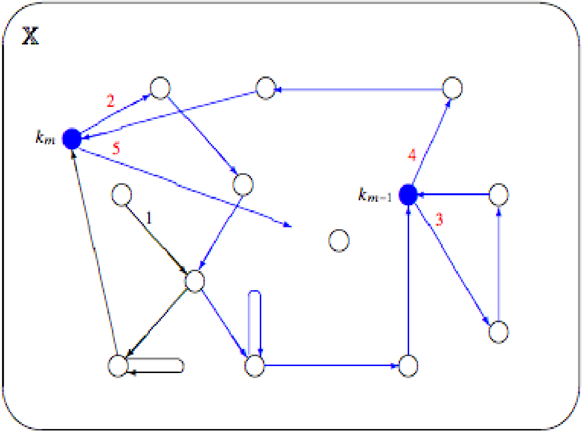
$(\mathcal{R})$: $p_{\varepsilon,ij} \to p_{0,ij}$ as $\varepsilon \to 0$, for $i, j \in \mathbb{X} = \{1, \ldots, m\}$.

$(\mathcal{S})$: $F_{\varepsilon,ij}(\cdot u_{\varepsilon,i}) \Rightarrow F_{0,ij}(\cdot)$ as $\varepsilon \to 0$, where **(a)** $F_{0,ij}(\cdot)$ are non-arithmetic distribution functions without singular component, for $i, j \in \mathbb{X}$ such that $p_{0,ij} > 0$, **(b)** $u_{\varepsilon,i} \in (0, \infty), \varepsilon \in (0, 1]$ and $u_{\varepsilon,i} \to u_{0,i}$ $\in (0, \infty]$ as $\varepsilon \to 0$, for $i \in \mathbb{X}$.

$(\mathcal{T})$: $u_{\varepsilon,i}^{-1} f_{\varepsilon,ij} = \int_0^\infty tF_{\varepsilon,ij}(dt) \to f_{0,ij} = \int_0^\infty tF_{0,ij}(dt)$ as $\varepsilon \to 0, i, j \in \mathbb{X}$.

$(\mathcal{U})$: $q_{\varepsilon,i}(s_\varepsilon u_{\varepsilon,i}, A) \to q_{0,i}(s, A)$ for any $0 \leq s_\varepsilon \to s$, as $\varepsilon \to 0$, and $s \in \mathrm{U}_A$, $A \in \Gamma, i \in \mathbb{X}$ where: (a) U_A is some Borel subset of $[0, \infty)$ such that the Lebesgue measure $m(\bar{\mathrm{U}}_A) = 0$, (b) $q_{0,i}(s, A)$ is a measurable function continuous almost everywhere with respect to the Lebesgue measure on $[0, \infty)$, (c) $\Gamma \subseteq \mathcal{B}_{\mathbb{Z}}$ and $\mathbb{Z} \in \Gamma$.

$(\mathcal{V})$: Non-zero transition probabilities $p_{\varepsilon,ij}$ and the initial normalisation functions $u_{\varepsilon,i}, i \in \mathbb{X}$ belong to a complete class of asymptotically comparable functions $\mathcal{H}$.



$$ {}_{\bar{k}_n}u_{\varepsilon,i} = \prod_{r=0}^{n}(1 - {}_{\bar{k}_r}p_{\varepsilon,ii})^{-1}u_{\varepsilon,i} $$

$$ \mathbf{(}\mathcal{W}\mathbf{)}\text{: } \lim_{\varepsilon\to 0}\frac{{}_{\bar{k}_n}u_{\varepsilon,k_{n+1}}}{{}_{\bar{k}_n}u_{\varepsilon,i}} = w_{0,k_{n+1},i} \in [0,\infty),\ i \in {}_{\bar{k}_n}\mathbb{X}, 0 \le n \le m-2. $$

$(\mathcal{X})$: $0 < {}_{\bar{k}_{m-2}}p_{\varepsilon,12},\ {}_{\bar{k}_{m-2}}p_{\varepsilon,21} \to 0$ as $\varepsilon \to 0$.

$(\mathcal{Y})$: ${}_{\bar{k}_{m-2}}p_{\varepsilon,12} / {}_{\bar{k}_{m-2}}p_{\varepsilon,21} \to \beta = {}_{\bar{k}_{m-2}}\beta \in (0, \infty)$ as $\varepsilon \to 0$.

$(\mathcal{Z})$: ${}_{\bar{k}_{m-2}}u_{\varepsilon,k_m} / {}_{\bar{k}_{m-2}}u_{\varepsilon,k_{m-1}} \to \gamma = {}_{\bar{k}_{m-2}}\gamma \in (0, \infty)$ as $\varepsilon \to 0$.

T *Let conditions $(\mathcal{R})$ – $(\mathcal{V})$, and $(\mathcal{W})$ – $(\mathcal{Z})$ (with some $\beta, \gamma \in (0, \infty)$) are satisfied. Then the following ergodic relation takes place for any $A \in \Gamma, i \in \mathbb{X}$ and any $0 \leq t_\varepsilon \to \infty$ as $\varepsilon \to 0$ such that $t_\varepsilon / {}_{\bar{k}_{m-2}}v_\varepsilon \to t \in (0, \infty)$ as $\varepsilon \to 0$,*

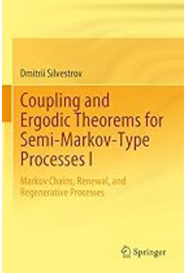
$$\mathsf{P}_{\varepsilon,i}(t_\varepsilon\, {}_{\bar{k}_{m-2}}u_\varepsilon, A) \to {}_{\bar{k}_{m-2}}\pi_{0,i}^{(\beta,\gamma)}(t, A) \text{ as } \varepsilon \to 0.$$

where the time compression factors functions ${}_{\bar{k}_{m-2}}v_\varepsilon$, ${}_{\bar{k}_{m-2}}u_\varepsilon$ and the stationary probabilities ${}_{\bar{k}_{m-2}}\pi_{0,i}^{(\beta,\gamma)}(t, A)$ are given by explicit recurrent formulas.

- 12 super-long-, long-, and short-time ergodic theorems for regularly and singularly perturbed multi-alternating regenerative processes are given.

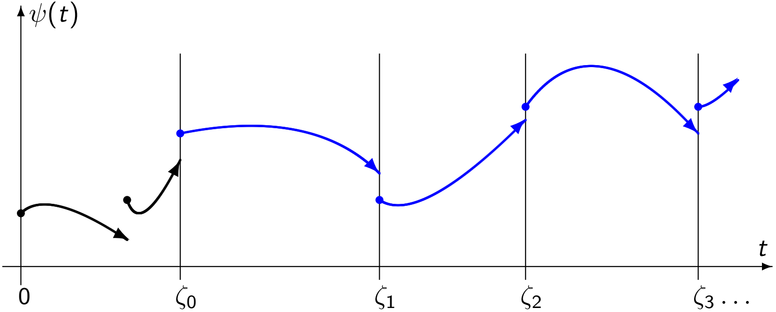
Gyllenberg and Silvestrov (2008), Silvestrov, D. and Silvestrov, S. (2017)

[2] Silvestrov, D. (2025). Coupling and Ergodic Theorems for Semi-Markov-Type Processes I: Markov Chains, Renewal and Regenerative Processes. Springer, Cham, xix+649 pp.



- Ergodic theorems with *explicit* power and exponential upper bounds for convergence rates for Markov chains, renewal and regenerative processes. These theorems are obtained using the *coupling method* in combination with the *method of test functions*.

Regenerative process $\psi(t), t \geq 0$ with the state space $\mathbb{X}$, regenerative moments $\zeta_n = \sum_{r=0}^{n} \xi_r$, $n = 0, 1, \ldots$ and the transition period $[0, \zeta_0)$.



$\bar{F}(u) = \mathsf{P}\{\xi_1 \leq u\}, u \geq 0$, $\bar{q}_t(A) = \mathsf{P}\{\psi(t) \in A, \xi_0 > t\}, A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$,

$\bar{\mathsf{P}}_t = \langle \bar{P}_t(A) = \mathsf{P}\{\psi(t) \in A\},\ A \in \mathcal{B}_{\mathbb{X}} \rangle, t \geq 0$,

$F(u) = \mathsf{P}\{\xi_1 \leq u\}, u \geq 0$, $q_t(A) = \mathsf{P}\{\psi(\zeta_0 + t) \in A, \xi_1 > t\}, A \in \mathcal{B}_{\mathbb{X}}, t \geq 0$,

$\mathsf{P}_t = \langle P_t(A) = \mathsf{P}\{\psi(\zeta_0 + t) \in A\},\ A \in \mathcal{B}_{\mathbb{X}} \rangle, t \geq 0$.

$$e_{k,\mathrm{F}} = \int_0^\infty u^k F(du), \, k \geq 1.$$

$\mathbf{E}_k$: $e_{k,\mathrm{F}} \in (0, \infty)$.

Condition $\mathbf{E}_1$ implies that the regenerative process $\psi(\cdot)$ has the statioanary distribution $\Pi = \langle \pi(A), A \in \mathcal{B}_{\mathbb{X}} \rangle$:

$$\pi(A) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \bar{P}_s(A) m(ds) = \frac{1}{e_{1,F}} \int_0^\infty q_s(A) m(ds), \; A \in \mathcal{B}_{\mathbb{X}}.$$

$$W(\mathrm{P}_1, \mathrm{P}_2) = \sup_{A \in \mathcal{B}_{\mathbb{X}}} |P_1(A) - P_2(A)|,$$

where $\mathrm{P}_i = \langle P_i(A), A \in \mathcal{B}_{\mathbb{X}} \rangle, i = 1, 2$.

$$W(\bar{\mathrm{P}}_t, \Pi) \to 0 \text{ as } t \to \infty,$$
$$W(\bar{\mathrm{P}}_t, \Pi) \leq G' t^{-g'} \to 0 \text{ as } t \to \infty,$$
$$W(\bar{\mathrm{P}}_t, \Pi) \leq G'' e^{-g'' t} \to 0 \text{ as } t \to \infty.$$

$$V_t(\mathsf{F}) = 1 - W(\mathsf{F}, \mathsf{F}_t) = 1 - \sup_{A \in \mathcal{B}_{[0,\infty)}} |\mathsf{F}_t(A) - \mathsf{F}(A)|.$$

where: (a) $\mathsf{F}_t = \langle F_t(A) = \mathsf{P}\{\xi_1 + t \in A\}, A \in \mathcal{B}_{[0,\infty)} \rangle$, $t \geq 0$, (b) $\mathsf{F} = \mathsf{F}_0$.

$$V_t(\mathsf{F}) = \sup_{\mathsf{F}_2(\cdot,\cdot) \in \mathcal{L}[\mathsf{F},\mathsf{F}]} \mathsf{P}\{\xi' = \xi'' + t\},$$

where: (a) $\mathcal{L}[\mathsf{F}, \mathsf{F}]$ is the family of all two-dimensional dictributions $F_2(u', u'')$ with marginals $F_2(u, \infty) = F_2(\infty, u) = F(u), u \geq 0$, (b) (ξ', ξ'') is a random vector with the distribution $\mathsf{P}\{\xi' \leq u', \xi'' \leq u''\}$ $= F_2(u', u''), u', u'' \geq 0$.

Let $q \in (0, 1)$ and $T > 0$:

$\mathbf{H}_{q,T}$: $V_t(\mathsf{F}) \geq q, t \in [0, T]$.

Distribution $\mathsf{F} = \langle F(u), u \geq 0 \rangle$ has a non-zero absolutely continuous component in the Lebesgue decomposition if and only if condition $\mathbf{H}_{q,T}$ is satisfied for some $q \in (0, 1)$ and $T > 0$.

$$F_\circ(u) = \frac{1}{e_{1,F}} \int_0^u (1 - F(s))ds, \ u \geq 0.$$

T$_1$ *Let conditions* $\mathbf{E}_1$ *and* $\mathbf{H}_{q,T}$ *hold. Then, for* $0 < t \to \infty$,

$$W(\bar{\mathsf{P}}_t, \mathsf{P}_t) \leq t^{-1}\big(A_{1,F} + B_{1,F} \int_0^t u\bar{F}(du) + B_{1,F}\, t(1 - \bar{F}(t)\big) \to 0,$$

$$W(\mathsf{P}_t, \Pi) \leq t^{-1}\big(A_{1,F} + B_{1,F} \int_0^t uF_\circ(du))du + B_{1,F}\, t(1 - F_\circ(t)\big) \to 0.$$

T$_2$ *Let conditions* $\mathbf{E}_k$ *and* $\mathbf{H}_{q,T}$ *hold. Then, for* $0 < t \to \infty$,

$$W(\bar{\mathsf{P}}_t, \mathsf{P}_t) \leq t^{-k}\big(A_{k,F} + \sum_{i=1}^k B_{k,i,F} \int_0^t u^i \bar{F}(du) + B_{k,k,F} t^k(1 - \bar{F}(t))\big) \to 0,$$

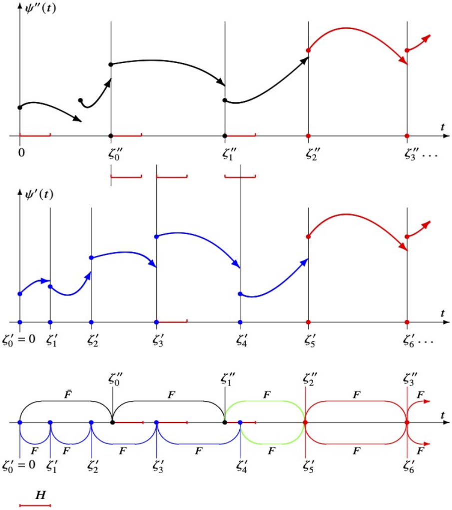
$$W(\mathsf{P}_t, \Pi) \leq t^{-k}\big(A_{k,F} + \sum_{i=1}^k B_{k,i,F} \int_0^t u^i F_\circ(du) + B_{k,k,F} t^k(1 - F_\circ(t))\big) \to 0.$$

$\mathbf{G}_\beta$: $E_{\beta,F} = \int_0^\infty e^{\beta u} F(du) \in (1, \infty)$ (for some $\beta > 0$).

T$_3$ *Let conditions* $\mathbf{G}_\beta$ *and* $\mathbf{H}_{q,T}$ *hold. Then, for* $\alpha \in [0, \alpha_{\beta,F})$ *and* $0 < t \to \infty$,

$$W(\bar{\mathsf{P}}_t, \mathsf{P}_t) \leq e^{-\alpha t} V_{\alpha,F}\big(1 + \int_0^t \alpha e^{\alpha u}(1 - \bar{F}(u))du\big) \to 0,$$

$$W(\mathsf{P}_t, \Pi) \leq e^{-\alpha t} V_{\alpha,F}\big(1 + \int_0^t \alpha e^{\alpha u}(1 - F_\circ(u))du\big) \to 0.$$


$\psi''(t)$
t
0
ζ''_0
ζ''_1
ζ''_2
$\zeta''_3 \ldots$
$\psi'(t)$
t
$\zeta'_0 = 0$
ζ'_1
ζ'_2
ζ'_3
ζ'_4
ζ'_5
$\zeta'_6 \ldots$
ζ''_0
ζ''_1
ζ''_2
ζ''_3
$\bar{F}$
F
t
$\zeta'_0 = 0$
ζ'_1
ζ'_2
ζ'_3
ζ'_4
ζ'_5
ζ'_6
H

The Markov renewal process $\bar{\psi} = \langle \psi_n = (\eta_n, \xi_n), n = 0, 1, \ldots \rangle$ is a homogeneous MC with a state space $\mathbb{X} \times [0, \infty)$ and transition probabilities $Q(x, A, u)$ $= \mathsf{P}\{\eta_{n+1} \in A, \xi_{n+1} \leq u | \eta_n = x, \xi_n = v\}, x \in \mathbb{X}, A \in \mathcal{B}_{\mathbb{X}}, u, v \geq 0$. The following relation defines the corresponding semi-Markov process,

$$\eta(t) = \eta_n \text{ for } t \in [\zeta_n, \zeta_{n+1}), n = 0, 1, \ldots, \text{ where } \zeta_n = \textstyle\sum_{k=1}^{n} \xi_k.$$

$$\tau(D) = \textstyle\sum_{k=1}^{\nu(D)} \kappa_k, \text{ where } \nu(D) = \min(n \geq 1 : \eta_n \in D),$$

$$M_r = \langle M_r(x) = \mathsf{E}_x \tau(D), x \in \mathbb{X} \rangle, \ r \geq 1.$$

$\mathcal{V} = \{v\}$ is the space of Borel measurable functions acting from $\mathbb{X} \to [0, \infty]$, and for $v_1, \ldots, v_k \in \mathcal{V}$,

$$\mathbf{P} v(x) = \mathsf{E}_x \kappa_1 \mathrm{I}(\eta_1 \in \bar{D}) v(\eta_1), x \in \mathbb{X},$$

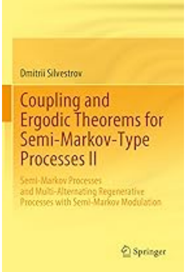
$$m_{[r]}(v_1, \ldots, v_{r-1}) = \langle \mathsf{E}_x \kappa_1^r + \textstyle\sum_{l=1}^{r-1} \binom{r}{l} \mathsf{E}_x \kappa_1^{r-l} \mathrm{I}(\eta_1 \in \bar{D}) v_l(\eta_1), x \in \mathbb{X} \rangle.$$

$\mathbf{T}_k$: There exist test functions $v_1, \ldots, v_k \in \mathcal{V}$ such that the following recursive test inequalitie hold, $v_r \geq m_{[r]}(v_1, \ldots, v_{r-1}) + \mathbf{P} v_r, r = 1, \ldots, k$.

T Moment functions $M_r, r = 1, \ldots, k$ are minimal in $\mathcal{V}$ solutions for the recursive integral equations, $M_r = m_{[r]}(M_1, \ldots, M_{r-1}) + \mathbf{P} M_r, r = 1, \ldots, k$, and $\mathbf{T}_k$ is the necessary and sufficient condition for holding the inequalities,

$$M_r \leq V_r = m_{[r]}(v_1, \ldots, v_{r-1}) + \mathbf{P} v_r \leq v_r, \ r = 1, \ldots, k.$$

[1] Silvestrov, D. (2025). Coupling and Ergodic Theorems for Semi-Markov-Type Processes II: Semi-Markov Processes and Multi-Alternating Regenerative Processes with Semi-Markov Modulation. Springer, Cham, xiv+550 pp.



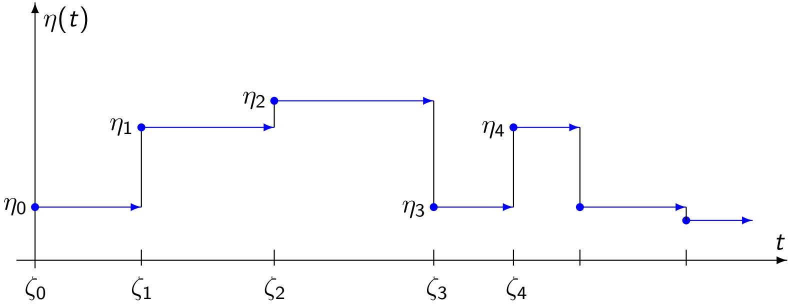
- Ergodic theorems with *explicit* power and exponential upper bounds for convergence rates for semi-Markov processes and multi-alternating regenerative processes with semi-Markov modulation. These theorems are obtained using the *coupling method* in combination with the *method of test functions* and the *method of artificial regeneration*.

The Markov renewal process $\bar{\psi} = \langle \psi_n = (\eta_n, \xi_n), n = 0, 1, \ldots \rangle$ is a homogeneous MC with a state space $\mathbb{X}$ and transition probabilities defined for $x \in \mathbb{X}, A \in \mathcal{B}_{\mathbb{X}}, u, v \geq 0$:

$$Q(x, A, u) = \mathsf{P}\{\eta_{n+1} \in A, \xi_{n+1} \leq u | \eta_n = x, \xi_n = v\},$$

The following relation defines the corresponding semi-Markov process,

$$\eta(t) = \eta_n \text{ for } t \in [\zeta_n, \zeta_{n+1}), n = 0, 1, \ldots, \text{ where } \zeta_n = \sum_{k=1}^{n} \xi_k.$$



A one-step splitting condition:

S$_1$: There exists a recurrent set $D \in \mathcal{B}_{\mathbb{X}}$ such that,

$$Q(x, A, u) = (1 - \varepsilon)\, Q_D(x, A, u) \\ + \varepsilon\, p_D(A)\, Q_{D,x}(u),\ A \in \mathcal{B}_{\mathbb{X}},\ u \in [0, \infty),\ \text{for } x \in D.$$

In this case, it is possible to define an extended Markov renewal process $\bar{\psi}^*_+ = \langle \psi^*_{+,n} = ((\chi^*_n, \eta^*_n), \xi^*_n), n = 0, 1, \ldots \rangle$ with the extended state space $\mathbb{Y}^* = \mathbb{Y} \times [0, \infty)$ (here, $\mathbb{Y} = \{0, 1\} \times \mathbb{X}$) and transition probabilities,

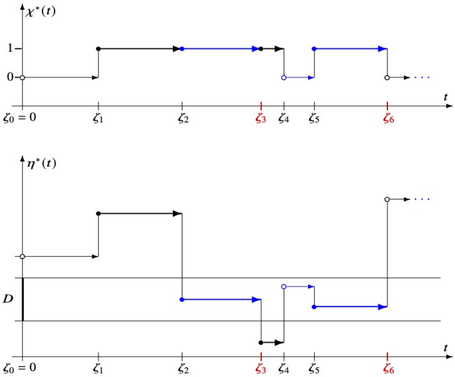
$$Q^*_D(\imath, x, \jmath, A, u) = \mathsf{P}\{\chi^*_1 = \jmath,\ {}_N\eta^*_1 \in A,\ {}_N\kappa^*_1 \le u \,|\, \chi^*_0 = \imath,\ \eta^*_0 = x\}$$

$$= \begin{cases} Q(x, A, u) q_\varepsilon(\jmath) & \text{for } \imath \in \{0, 1\}, x \in \bar{D}, \\ & \jmath \in \{0, 1\}, A \in \mathcal{B}_{\mathbb{X}}, u \in [0, \infty), \\ Q_D(x, A, u) q_\varepsilon(\jmath) & \text{for } \imath = 0, x \in D, \\ & \jmath \in \{0, 1\}, A \in \mathcal{B}_{\mathbb{X}}, u \in [0, \infty), \\ p_D(A)\, Q_{D,x}(u) q_\varepsilon(\jmath) & \text{for } \imath = 1, x \in D, \\ & \jmath \in \{0, 1\}, A \in \mathcal{B}_{\mathbb{X}}, u \in [0, \infty), \end{cases}$$

where $\bar{q}_\varepsilon$ is the "tossing coin" distribution,

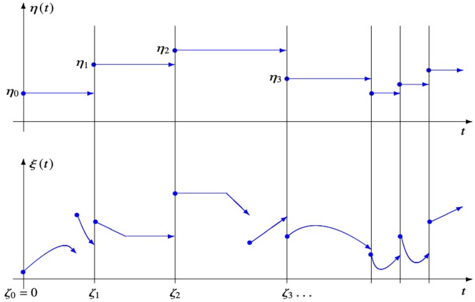
$$\bar{q}_\varepsilon = \langle q_\varepsilon(\jmath) = (1 - \varepsilon)\mathrm{I}(\jmath = 0) + \varepsilon \mathrm{I}(\jmath = 1), \jmath \in \{0, 1\} \rangle.$$

V2: Artificial Regeneration for SMP



$\imath = 0,\ x \in \bar{D}$	$Q(x,\cdot,\cdot)q_\varepsilon(\cdot)$	
$\imath = 1,\ x \in \bar{D}$	$Q(x,\cdot,\cdot)q_\varepsilon(\cdot)$	
$\imath = 0,\ x \in D$	$Q_D(x,\cdot,\cdot)q_\varepsilon(\cdot)$	
$\imath = 1,\ x \in D$	$p_D(\cdot)Q_{D,x}(\cdot)q_\varepsilon(\cdot)$	

V2: Multi-alernating Regenerative Processes with Semi-Markov Modulation



- Ch.1: Introduction (examples, models, results)
- Ch. 2: Summary of ergodic theorems for regenerative processes,
- Ch. 3: Extensions of hitting times and method of test functions
- Ch. 4: Birth-death processes
- Ch. 6: Discrete semi-Markov processes
- Ch. 7: Queuing systems
- Ch. 8: Semi-Markov processes with atoms
- Ch. 9: Semi-Markov processes with distributional atoms
- Ch. 10 -11: Semi-Markov processes and artificial regeneration
- Ch. 11-12: Multi-Alternating regenerative processes with semi-Markov modulation

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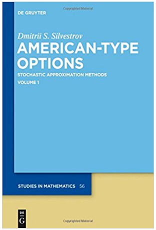

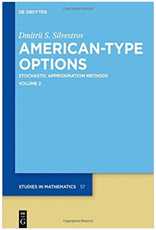

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THANK YOU FOR YOUR ATTENTION !