Estimation of risk indicators for (hidden) semi-Markov models

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Markov, Semi-Markov Models and Associated Fields
(from theory to application and back)

Outline

- Semi-Markov chains (SMCs)
  - Motivation
  - Introduction
- Risk indicators for SMCs
  - Conditional Mean time to failure
    $\hookrightarrow$ Application : Wind energy production
- Risk indicators for hidden SMCs
  - Rate of occurrence of failures
- Risk indicators for SM processes
- Ongoing work & perspectives

Motivation

Context

- How could we estimate, prevent and manage the risk of failures for random systems ?
- Which stochastic models should be used to describe such systems ?

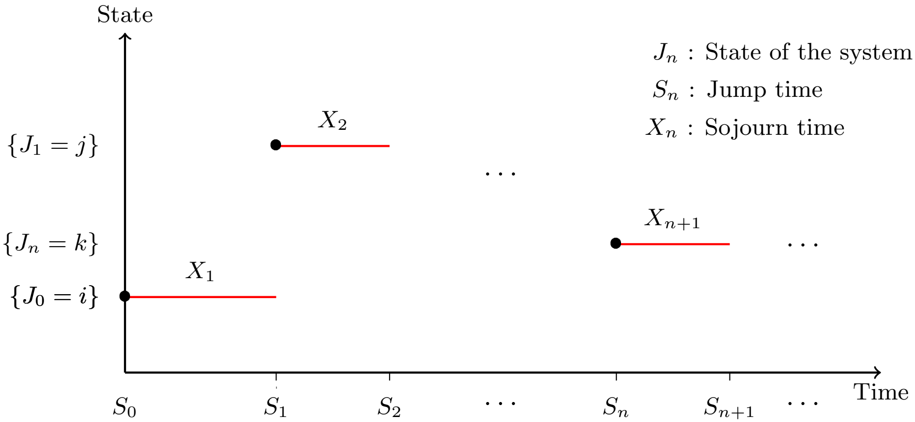
  $\hookrightarrow$ Poisson, Markov, Cox, Auto-regressive Markov switching, ...

Objectives

- Random systems description by semi-Markov models.
- Estimation of empirical reliability indicators

  $\hookrightarrow$ ROCOF, reliability, availability, MTTF, MTBF, ...

Semi-Markov models



Definition

The chain $(\mathbf{J}, \mathbf{S}) = (J_n, S_n)_{n \in \mathbb{N}}$ *is a Markov renewal chain if it satisfies*

$$\forall k, n \in \mathbb{N},\ \forall i, j \in E, \quad \mathbb{P}(J_{n+1} = j, X_{n+1} = k | S_0, \ldots, S_n; J_0, \ldots, J_n = i) \\ = \mathbb{P}(J_{n+1} = j, X_{n+1} = k | J_n = i).$$

Definition

The semi-Markov chain (SMC) $\mathbf{Z} = (Z_k)_{k \in \mathbb{N}}$ *is defined by*

$$Z_k = J_{N(k)}, \text{ where } N(k) = \max\{n \in \mathbb{N} | S_n \leq k\}.$$

.

The chain $\mathbf{J} = (J_n)_{n\in\mathbb{N}}$ (*embedded Markov chain*) is characterized by

- its initial law $\boldsymbol{\alpha} = (\alpha(i);\ i \in E)$, where

$$\alpha_i = \mathbb{P}(J_0 = i),$$

- its transition matrix $\mathbf{P} = (p_{ij};\ i,j \in E)$, where

$$p_{ij} = \mathbb{P}(J_{n+1} = j | J_n = i), \quad \forall n \in \mathbb{N}.$$

- Semi-Markov kernel

$$\mathbf{q} = (q_{ij}(k);\ i,j \in E, k \in \mathbb{N}), \quad \text{where} \quad q_{ij}(k) = \mathbb{P}(J_{n+1} = j, X_{n+1} = k \mid J_n = i),$$

- Conditional sojourn time distribution

$$\mathbf{f} = (f_{ij}(k);\ i,j \in E, k \in \mathbb{N}), \quad \text{where} \quad f_{ij}(k) = \mathbb{P}(X_{n+1} = k \mid J_n = i, J_{n+1} = j).$$

Conditional mean time to failure

Wind energy production (ANR CAESARS [1])

Motivation

- Provide support for wind farms' electrical power production management.
- Estimate failure risks for wind farms via semi-Markov models (D'Amico et al., 2015).

Main results

- Proposition of a risk indicator : the Conditional Mean Time To Failure (CMTTF)
- Statistical estimation & application to wind data

[1] https://caesars.math.cnrs.fr/

Conditional mean time to failure

Z takes its values in $E = \{1, 2, \ldots, s\}$ such that $E = U \cup D$ $(U, D \neq \emptyset)$ with

- $U = \{1, 2, \ldots, r\}$ $\hookrightarrow$ up states
- $D = \{r+1, \ldots, s\}$ $\hookrightarrow$ down states
- The first passage time in D is defined by:

$$T_D = \inf\{k \in \mathbb{N} : Z_k \in D\} \quad \text{and} \quad \inf \varnothing = \infty.$$

Definition

The conditional mean time to failure is defined by

$$CMTTF_i = \mathbb{E}(T_D | J_0 = i),$$

for any state $i \in U$.

Evaluation

Vector of the conditional mean times to failure (Barbu and Limnios, 2008):

$$\begin{aligned}\mathbf{CMTTF} &= (CMTTF_1, \ldots, CMTTF_r)^\top \\ &= (\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{m_1},\end{aligned}$$

where $\mathbf{P}_{11} = (p_{ij};\ i,j \in U)$ and $\mathbf{m}_1 = (\mathbb{E}(S_1 | J_0 = i);\ i \in U)$.

Empirical estimation

Given a trajectory of length M, $H(M)$, the estimator is

$$\widehat{\mathbf{CMTTF}}(M) = (\mathbf{I} - \widehat{\mathbf{P}}_{11}(M))^{-1}\widehat{\mathbf{m}}_1(M),$$

where

- $\widehat{\mathbf{P}}_{11}(M) = (\widehat{p}_{ij}(M);\ i,j \in U)$, with $\widehat{p}_{ij}(M) = \frac{N_{ij}(M)}{N_i(M)}$,
- $\widehat{\mathbf{m}}_1(M) = (\widehat{m}_i(M); i \in U)^\top$, with $\widehat{m}_i(M) = \sum_{\ell \geq 0} \widehat{\overline{H}}_i(\ell, M)$.

Theorem (V. and Brouste, 2019)

For any $i \in U$, $C\widehat{MTTF}_i(M)$ is strongly consistent, i.e.

$$C\widehat{MTTF}_i(M) \xrightarrow[M\to\infty]{a.s.} CMTTF_i.$$

Theorem (V. and Brouste, 2019)

For any $i \in U$, $C\widehat{MTTF}_i(M)$, is asymptotically normal, i.e

$$\sqrt{M}(C\widehat{MTTF}_i(M) - CMTTF_i) \xrightarrow[M\to\infty]{\mathcal{L}} \mathcal{N}(0, \sigma^2_{CMTTF_i}),$$

with

$$\sigma^2_{CMTTF_i} = \sum_{m\in E} a^2_{im}\mu_{mm}\Big(\sigma^2_m + \sum_{\ell\in E}(\eta_\ell - \widetilde{\eta}_m)^2 p_{m\ell} + 2\sum_{\ell\in E}\eta_\ell Q_{m\ell}\Big),$$

where $Q_{m\ell} = \sum_{u=1}^{+\infty}(u - m_m)q_{m\ell}(u)$, $a_{ij} = (\mathbf{I} - \mathbf{P}_{11})^{-1}_{ij}$, $\eta_\ell = \sum_{r\in U} m_r a_{r\ell}$, $\widetilde{\eta}_m = \sum_{j\in U} p_{mj}\eta_j$, *and* σ^2_m *is the variance of the sojourn time in state* m.

Semi-Markov modeling

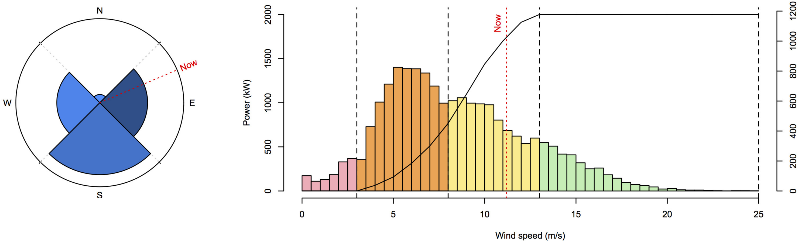


Figure: Wind directions and speeds states. The transfer power function of a 2MW wind turbine is superposed on the wind speeds histogram.

Data Selection

- Real Data $\hookrightarrow$ EREN Group
- Simulated Data $\hookrightarrow$ NRE Lab

States

- $U \hookrightarrow (Direction, Speed)$, $Speed > 3\ m/s$.
- $D \hookrightarrow (Direction, Speed)$, $Speed \leq 3\ m/s$.

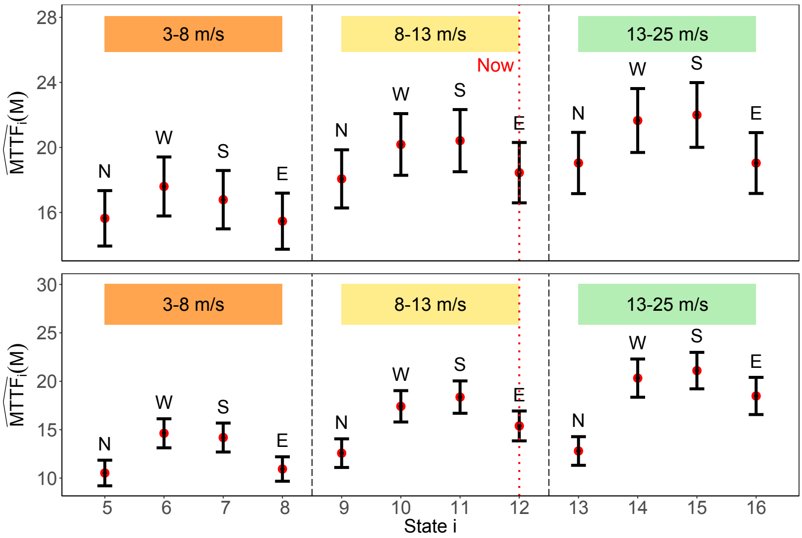
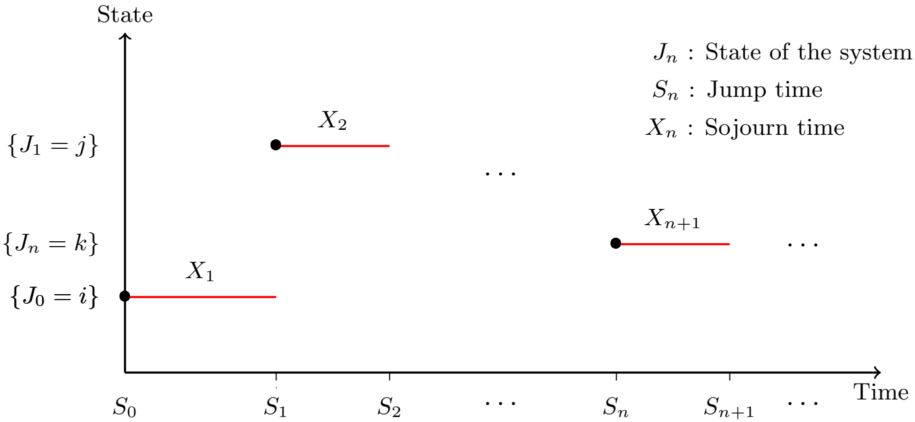


Figure: Empirical estimators of the conditional mean times to failure (in hours), $\widehat{MTTF}_i(M)$, $i \in U$, evaluated for real data (EREN) (upper panel) and simulated data (NREL) (lower panel).

...from SMCs to hidden semi-Markov chains

Semi-Markov chains



Hidden Markov renewal chains

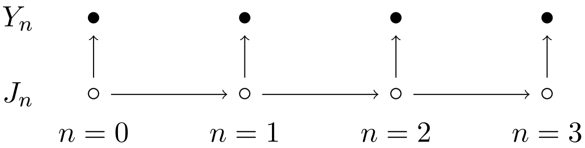


Figure: Trajectory of a hidden Markov renewal chain.

- *Underlying chain* $\hookrightarrow$ semi-Markov chain $\mathbf{Z} = (Z_k)_{k\in\mathbb{N}}$
- *Observation sequence* $\hookrightarrow$ $\mathbf{Y} = (Y_n)_{n\in\mathbb{N}}$
- ⚠ *Assumption:* The process is observed only at jump times.

Definitions

- The process $(\mathbf{J}, \mathbf{S}, \mathbf{Y}) = (J_n, S_n, Y_n)_{n\in\mathbb{N}}$ is a hidden Markov renewal chain.
- State spaces
  - Hidden Markov renewal chain $(\mathbf{J}, \mathbf{S}, \mathbf{Y}) = (J_n, S_n, Y_n)_{n\in\mathbb{N}} \hookrightarrow E^*$
  - Markov renewal chain $(\mathbf{J}, \mathbf{S}) = (J_n, S_n)_{n\in\mathbb{N}} \hookrightarrow L = E \times \mathbb{N}$
  - Observation process $\mathbf{Y} = (Y_n)_{n\in\mathbb{N}} \hookrightarrow A$
- Emission matrix $\mathbf{R} = (R_{i;a};\ i \in E, a \in A)$, where

$$R_{i;a} = \mathbb{P}(Y_n = a | J_n = i), \quad \forall n \in \mathbb{N}.$$

Rate of occurrence of failures

Literature

- Markov models (continuous time)
  - Discrete state space: Yeh (1997), D'Amico (2015).
- Semi-Markov models (continuous time)
  - Discrete state space: Ouhbi and Limnios (2002).
  - Continuous state space: Limnios (2012).
  - Application fields: **Seismology, Finance, Biology** etc
- (Hidden) semi-Markov models (discrete time)
  - Discrete state space: V. et al. (2014), V. and Limnios (2015).
  - Application fields: **Seismology, Finance, Biology** etc

Rate of occurrence of failures

- The observation space $A = \{1, 2, \dots, s\}$ is partitioned into:
  - $U = \{1, 2, \dots, r\} \quad \hookrightarrow$ up observations
  - $D = \{r+1, \dots, s\} \hookrightarrow$ down observations
- At time $k \in \mathbb{N}^*$, the number of transitions of the observation sequence from U to D is:

$$N_U(k) = \sum_{\ell=1}^{k} 1_{\{Y_{\ell-1} \in U, Y_\ell \in D\}}.$$

ROCOF

The rate of occurrence of failures is the mean number of observed failures at time $k \in \mathbb{N}^*$:

$$r_U(k) = \mathbb{E}[N_U(k) - N_U(k-1)].$$

Theorem (V. and Limnios, 2015)

The ROCOF at time $k \in \mathbb{N}^*$ is given by

$$r_U(k) \;=\; \sum_{(i,y)\in E\times A} \sum_{\ell=1}^{k} \sum_{s_0,s_1\in E} \sum_{k_0=0}^{k} a(i,0) R_{i;y} \mathrm{R}_{s_0}(U) \mathrm{R}_{s_1}(D) q_{is_0}^{(l-1)}(k_0) q_{s_0 s_1}(k-k_0).$$

Estimation

Given a trajectory of the HMRC up to fixed arbitrary time $M \in \mathbb{N}$, the estimator of ROCOF at time $k \in \mathbb{N}^*$ is given by

$$\widehat{r}_U(k,M) = \sum_{(i,y)\in E\times A} \sum_{\ell=1}^{k} \sum_{s_0,s_1\in E} \sum_{k_0=0}^{k} \widehat{a}(i,M) \widehat{R}_{i;y}(M) \widehat{\mathrm{R}}_{s_0}(U,M) \widehat{\mathrm{R}}_{s_1}(D,M) \widehat{q}_{is_0}^{(l-1)}(k_0,M) \widehat{q}_{s_0 s_1}(k-k_0,M).$$

Proposition (V. and Limnios, 2015)

For any fixed, arbitrary $k \in \mathbb{N}^*$, $\widehat{r}_U(k, M)$ *is strongly consistent, i.e.*

$$\widehat{r}_U(k, M) \xrightarrow[M\to\infty]{a.s.} r_U(k).$$

Theorem (V. and Limnios, 2015)

For any fixed, arbitrary $k \in \mathbb{N}^*$,

$$\sqrt{M}\big(\widehat{r}_U(k, M)\big) \xrightarrow[M\to\infty]{\mathcal{L}} \mathcal{N}(0, \Phi'\Gamma\Phi'^{\top}), \quad \textit{with}$$

$$\Phi\Big((R_{j';m'}, q_{i'j'}(k'-k_0')); (i', k_0') \in L, (i', j', k') \in E^*\Big)$$
$$= \sum_{i\in U}\sum_{j\in D}\sum_{(\ell,y)\in E\times A}\sum_{l=1}^{k}\sum_{s_0,s_1\in E}\sum_{k_0=0}^{k} a(\ell, y)R_{\ell;y}R_{s_0}(i)R_{s_1}(j)q_{is_0}^{(l-1)}(k_0)q_{s_0s_1}(k-k_0),$$

and Γ *the covariance matrix of* $\sqrt{M}(\widehat{q}_{ij}(t_2-t_1, M)\widehat{R}_j(m, M) - q_{ij}(t_2-t_1)R_j(m))$.

Hidden semi-Markov chains

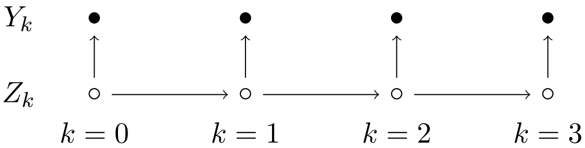


Figure: Trajectory of a hidden semi-Markov chain (HMRC).

- *Underlying chain* $\hookrightarrow$ semi-Markov chain $\mathbf{Z} = (Z_k)_{k \in \mathbb{N}}$
- *Observation sequence* $\hookrightarrow$ $\mathbf{Y} = (Y_k)_{k \in \mathbb{N}}$
- ~~*Assumption:*~~ The process is observed only at jump times

Coupled Markov chain

Definition

The sequence of the backward recurrence times is defined by $\mathbf{U} = (U_k)_{k\in\mathbb{N}}$, *where* $U_k = k - S_{N(k)}$.

Theorem (Limnios and Oprişan, 2001)

The process $(\mathbf{Z}, \mathbf{U}) = (Z_k, U_k)_{k\in\mathbb{N}}$ *is the coupled Markov chain with initial law*

$$\widetilde{\boldsymbol{a}} = (\widetilde{a}(i,0);\ i \in E),$$

and transition matrix

$$\widetilde{\mathbf{P}} = (\widetilde{p}\big((i,t_1),(j,t_2)\big);\ (i,t_1),(j,t_2) \in E \times \mathbb{N}).$$

Theorem (V., 2019)

The ROCOF at time $k \in \mathbb{N}^*$ is given by

$$r(k) = \sum_{a_1 \in U} \sum_{a_2 \in D} \sum_{i \in E} \sum_{j \in E} \sum_{u_1 \in T_{k-1}} \sum_{u_2 \in T_k} R_{j;a_2} \widetilde{p}\big((i,u_1),(j,u_2)\big) R_{i;a_1} (\widetilde{\mathbf{a}}\widetilde{\mathbf{P}})^{k-1}(i,u_1),$$

where by Chryssaphinou et al. (2008)

$$\widetilde{p}\big((i,t_1),(j,t_2)\big) = \begin{cases} q_{ij}(t_1+1)/\overline{H}_i(t_1), & \text{if} \quad i \neq j,\ t_2 = 0, \\ \overline{H}_i(t_1+1)/\overline{H}_i(t_1), & \text{if} \quad i = j,\ t_2 - t_1 = 1, \\ 0, & \text{elsewhere}, \end{cases}$$

with $\overline{H}_i(k) = \mathbb{P}(X_{l+1} > k | J_l = i)$ and $T_k = \{1, \ldots, k\}$.

Other results

- ✓ Statistical estimation (plug-in MLE)
- ✓ Asymptotic properties

...from SMCs to semi-Markov processes

Context

- $(J_n)_{n\in\mathbb{N}}$ is defined in a discrete state space E;
- $(S_n)_{n\in\mathbb{N}}$ is defined in $\mathbb{R}^+$.

Definition

- *The Markov renewal process* $(J_n, S_n)_{n\in\mathbb{N}}$ *satisfies a.s.*

$$\mathbb{P}(J_{n+1}, X_{n+1} \leq x | J_0, \ldots, J_n, S_0, \ldots, S_n) = \mathbb{P}(J_{n+1}, X_{n+1} \leq x | J_n),$$

$\forall n \in \mathbb{N}$ *and* $\forall x \in \mathbb{R}^+$.

- *The semi-Markov process is defined by* $Z_t = J_{N(t)}$, *where*

$$N(t) = \sup\{n \in \mathbb{N} : S_n \leq t\},\ t \in \mathbb{R}^+.$$

Semi-Markov kernel

$$Q_{ij}(x) = \mathbb{P}(J_{n+1} = j, X_{n+1} \leq x | J_n = i), \quad \forall i, j \in E, x \in \mathbb{R}^+$$

Conditional sojourn time distribution

$$F_{ij}(x) = \mathbb{P}(X_{n+1} \leq x | J_n = i, J_{n+1} = j), \quad \forall i, j \in E, x \in \mathbb{R}^+$$

Survival time distribution

$$H_i(x) = \mathbb{P}(X_{n+1} \geq x | J_n = i), \quad \forall i \in E, x \in \mathbb{R}^+$$

Bootstrapped estimators

- Let $\mathbf{W} \equiv (W_{nj}, j = 1, \ldots, n, n = 1, 2 \ldots)$ be an array of random variables.
- Semi-Markov kernel (Bouzebda et al., 2018)

$$\widetilde{Q}^{W}_{ij}(x, T) \quad := \quad \frac{1}{N_i(T)} \sum_{k=1}^{N(T)} W_{N(T)k} \mathbb{1}_{\{J_{k-1}=i, J_k=j\}}, \quad 0 \leq x \leq T, \quad i, j \in E.$$

- Conditional sojourn time distribution (Bouzebda et al., 2018)

$$\widetilde{F}^{W}_{ij}(x, T) \quad := \quad \frac{1}{N_{i,j}(T)} \sum_{k=1}^{N(T)} W_{N(T)k} \mathbb{1}_{\{J_{k-1}=i, J_k=j\}}, \; 0 \leq x \leq T, \; i, j \in E.$$

Reliability

- For $x \in \mathbb{R}^+$, the reliability is defined by

$$R(x) = \mathbb{P}(Z_u \in U,\ \forall u \in [0, x]).$$

- The bootstrapped estimator (V. and Bouzebda, 2024) is defined by

$$\widetilde{R}^W(x,T) = a(0)(\mathtt{I} - \widetilde{Q}^W_{00}(x,T))^{(-1)} * (\mathtt{I} - \widetilde{H}^W_0(x,T))1_r, \quad \text{for any } 0 < x < T.$$

Theorem (V. and Bouzebda, 2024)

Under some assumptions[2]*, for any fixed $x \in \mathbb{R}^+$, we have*

$$T^{1/2}\big(\widehat{R}^W(x,T) - R(x)\big) \xrightarrow[T\to\infty]{\mathcal{L}} N\big(0, c^2\sigma^2(x)\big).$$

[2] Mason and Newton (1992), Præstgaard and Wellner, (1993)

Moments of the time to failure

- ongoing work -

in collaboration with Samis Trevezas

Mean time to failure

The partition of the transition matrix of the coupled MC into up ($\overline{\mathrm{U}} = U \times [1, m]$) and down states ($\overline{\mathrm{D}} = D \times [1, m]$) is given by

$$
\begin{array}{cc}
 & \begin{array}{cc} \overline{\mathrm{U}} & \overline{\mathrm{D}} \end{array} \\
\widetilde{\mathbf{P}} = & \begin{pmatrix} \widetilde{\mathbf{P}}_{\mathrm{UU}} & \widetilde{\mathbf{P}}_{\mathrm{UD}} \\ \widetilde{\mathbf{P}}_{\mathrm{DU}} & \widetilde{\mathbf{P}}_{\mathrm{DD}} \end{pmatrix} \begin{array}{c} \overline{\mathrm{U}} \\ \overline{\mathrm{D}} \end{array} .
\end{array}
$$

Proposition

If the coupled MC is irreducible (and positive reccurent), then the mean time to failure is given by

$$
\begin{aligned}
MTTF &= \boldsymbol{\alpha}_{\mathrm{U}} \left(\mathbf{I} - \mathbf{P}_{\mathrm{UU}}\right)^{-1} \mathbf{m}_{\mathrm{U}} \\
&= \boldsymbol{\alpha}_{\overline{\mathrm{U}}} \left(\mathbf{I} - \widetilde{\mathbf{P}}_{\mathrm{UU}}\right)^{-1} \mathbf{1}^{\top}.
\end{aligned}
$$

Proposition

If the coupled MC is irreducible, the k-th order ($k \geq 2$) factorial moment of the time to failure is given by

$$\begin{aligned} M_k &= \mathbb{E}(T_{\mathrm{D}}(T_{\mathrm{D}}-1)\cdots(T_{\mathrm{D}}-k+1)) \\ &= \sum_{n=k}^{+\infty} n(n-1)\cdots(n-k+1)\mathbb{P}(T_{\mathrm{D}}=n) \\ &= k!\widetilde{\mathbf{P}}_{\mathrm{UU}}^{\mathbf{k-1}}\left(\mathbf{I}-\widetilde{\mathbf{P}}_{\mathrm{UU}}\right)^{-\mathbf{k}}. \end{aligned}$$

Proposition

If the coupled MC is irreducible, the variance of the time to failure is given by

$$VTTF = \boldsymbol{\alpha}_{\overline{\mathrm{U}}}\left(2\left(\mathbf{I}-\widetilde{\mathbf{P}}_{\mathrm{UU}}\right)^{-2}-\left(\mathbf{I}-\widetilde{\mathbf{P}}_{\mathrm{UU}}\right)^{-1}\right)1^{\top}-\left(\boldsymbol{\alpha}_{\overline{\mathrm{U}}}\left(\mathbf{I}-\widetilde{\mathbf{P}}_{\mathrm{UU}}\right)^{-1}1^{\top}\right)^{2}.$$

Statistical estimation

- the case of multiple trajectories -

Statistical estimation

We consider K trajectories of the SMC of fixed length M.

- $N_{i,u;j}(K)$: number of transitions from the state (i,u) to the state $(j,\cdot)$.
- $N_{i,u}(K)$: number of visits to the state (i,u).

Empirical estimators

The estimators of the transition probabilities of the coupled MC are given by

$$\begin{aligned}\widehat{p}_{i,u;j} &= \frac{N_{i,u;j}(K)}{N_{i,u}(K)}, \qquad (i,u)\in E\times\mathbb{N}_{\le M-1},\ j\in E,\\ \widehat{\alpha}_{i,0} &= \frac{N_{i,0}(K)}{K}, \qquad i\in E,\end{aligned}$$

where $0/0=0$ by convention.

$\hookrightarrow$For consistency and asymptotic normality see Trevezas and Limnios (2011).

Consistency and asymptotic normality for MTTF

Theorem

As $K \longrightarrow \infty$, we have

- $\left(\widehat{MTTF}(K) - MTTF\right) \xrightarrow[K\to\infty]{a.s} 0,$
- $\sqrt{K}\left(\widehat{MTTF}(K) - MTTF\right)$ *converges in distribution to a centered Gaussian random variable, i.e.*

$$\sqrt{K}\left(\widehat{MTTF}(K) - MTTF\right) \xrightarrow[K\to\infty]{\mathcal{L}} \mathcal{N}(0, \sigma_m^2).$$

Consistency and asymptotic normality for VTTF

Theorem

As $K \longrightarrow \infty$, we have

- $\left(\widehat{VTTF}(K) - VTTF\right) \xrightarrow[K\to\infty]{a.s} 0,$
- $\sqrt{K}\left(\widehat{VTTF} - VTTF\right)$ *converges in distribution to a centered Gaussian random variable, i.e.*

$$\sqrt{K}\left(\widehat{VTTF}(K) - VTTF\right) \xrightarrow[K\to\infty]{\mathcal{L}} \mathcal{N}(0, \sigma_v^2).$$

Consistency and asymptotic normality for factorial moments

Theorem

For any $k \geq 2$, as $K \longrightarrow \infty$, we have

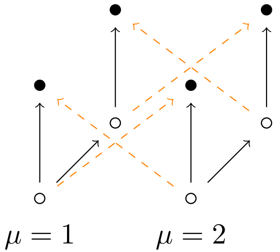
- $\left(\widehat{\boldsymbol{M}}_k - \boldsymbol{M}_k\right) \xrightarrow[K\to\infty]{a.s} 0$,
- $\sqrt{K}\left(\widehat{\boldsymbol{M}}_k - \boldsymbol{M}_k\right)$ *converges in distribution to a centered Gaussian random variable, i.e.*

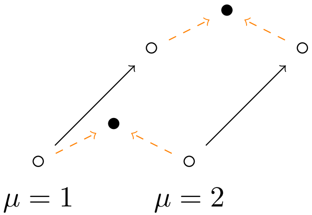
$$\sqrt{K}\left(\widehat{\boldsymbol{M}}_k - \boldsymbol{M}_k\right) \xrightarrow[K\to\infty]{\mathcal{L}} \mathcal{N}(0, \Sigma_{\boldsymbol{M}_k}).$$

Perspectives

Risk indicators are still unexplored for

- Countably infinite SMCs
- Hidden SMCs
- Multi-chain HMMs





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MERCI!

Appendix

- **Asymptotic variance**:

$$\sigma_{ij}^2(x) = \sum_{i\in E}\sum_{j\in E}\mu_{ii}\Bigg\{\Bigg(B_{ij}^U - \mathbb{1}_{\{i\in U\}}\sum_{r\in U}\alpha(r)\psi_{ri}\Bigg)^2 * q_{ij}(x)$$
$$- \Bigg[\Bigg(B_{ij}^U * q_{ij} - \mathbb{1}_{\{i\in U\}}\sum_{r\in U}\alpha(r)\psi_{ri} * Q_{ij}\Bigg)\Bigg]^2(x)\Bigg\},$$

where

$$B_{ij}^U = \sum_{n\in U}\sum_{k\in U}\alpha(n)\psi_{ni} * \psi_{jk} * \big(I - \mathrm{diag}(Q\mathbf{1})\big)_{kk}, \quad i,j\in E.$$

- **Mean reccurrence times of** (S_n^i):

$$\mu_{ii} = \mathbb{E}(S_2^i - S_1^i), \quad i\in E.$$

- **Matrix convolution product**:

  If

  - $\mathbf{A}(\cdot) := (A_{ij}(\cdot); 1 \leq i \leq n, 1 \leq j \leq m)$ a matrix-valued function
  - $\mathbf{B}(\cdot) := (B_{ij}(\cdot); 1 \leq i \leq m, 1 \leq j \leq r)$ a matrix-valued function

  Then, $\mathbf{A} * \mathbf{B}(\cdot)$ is the matrix-valued function $\mathbf{C}(\cdot) := (C_{ij}(\cdot); 1 \leq i \leq n, 1 \leq j \leq r)$, where

$$C_{ij}(k) = \sum_{s=1}^{m} \sum_{l=0}^{k} A_{is}(l) B_{sj}(k-l), \quad k \in \mathbb{N}, \quad i, j \in E.$$